MATH 208, EXAM 1

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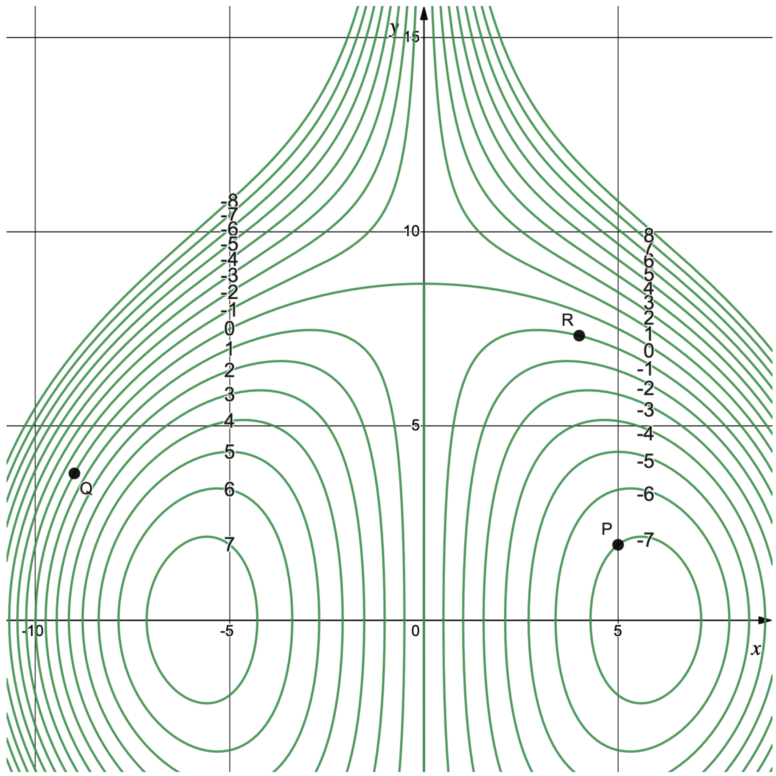
Name: Solutions NUID: ____________

Instructions.

- You should have 9 pages on which 7 problems are printed.
- You have 50 minutes: the exam will begin on the hour and end promptly, 50 minutes later.
- Show all work unless otherwise specified. What you write on the page must convince me that you understand the problem and its solution.
- Read each problem carefully.
- You do not need to simplify your answers, unless the instructions for a problem indicate otherwise.
- You are not allowed a calculator, notes, textbooks, or access to any electronic devices.
- Don't panic. Good luck!

Date: Fall 2022.

Below is pictured a contour map of a function $f(x, y)$. (Notice that some labels for contours appear in the left half of the picture, and others appear in the right half.) Problem 1 refers to this graph.



Problem 1 (2 points each)**.** Answer each question about the partial derivatives of f at the point Q.

(a) At the point Q, f_x is . . .

- [x] POSITIVE
- [ ] NEGATIVE
- [ ] APPROXIMATELY 0

(b) At the point Q, f_y is . . .

- [ ] POSITIVE
- [x] NEGATIVE
- [ ] APPROXIMATELY 0

(c) At the point Q, f_{xx} is . . .

- [ ] POSITIVE
- [x] NEGATIVE
- [ ] APPROXIMATELY 0

← f is concave down in each direction

(d) At the point Q, f_{yy} is . . .

- [ ] POSITIVE
- [x] NEGATIVE
- [ ] APPROXIMATELY 0

(e) Which of the graphs below is the correct graph of the x-trace $f(x, 1.9)$? (The point P is $(5, 1.9)$.)

- [ ] graph (i)
- [ ] graph (ii)
- [x] graph (iii)
- [ ] graph (iv)
- [ ] graph (v)

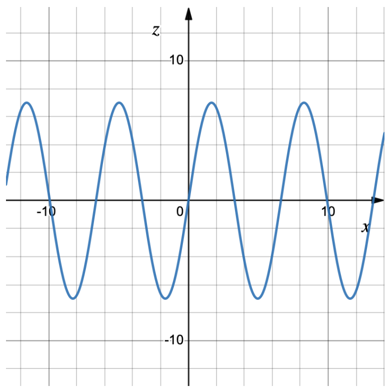
Notice:

$f(-5, 1.9) = 7$

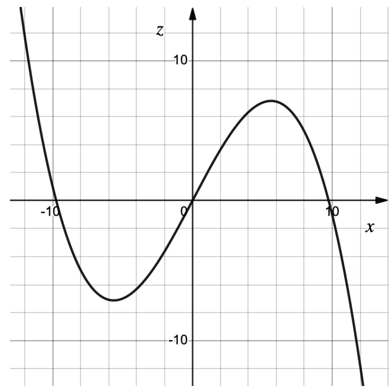
$f(0, 1.9) = 0$

$f(5, 1.9) = 7$

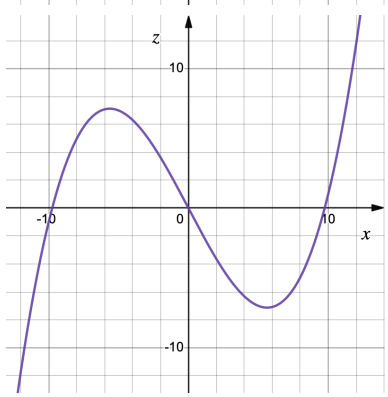
(i)



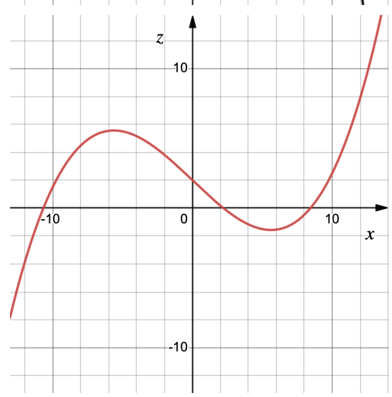
(ii)



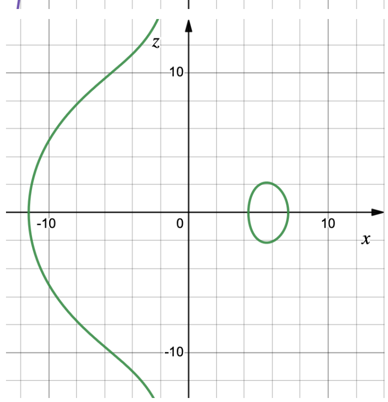
(iii)



(iv)



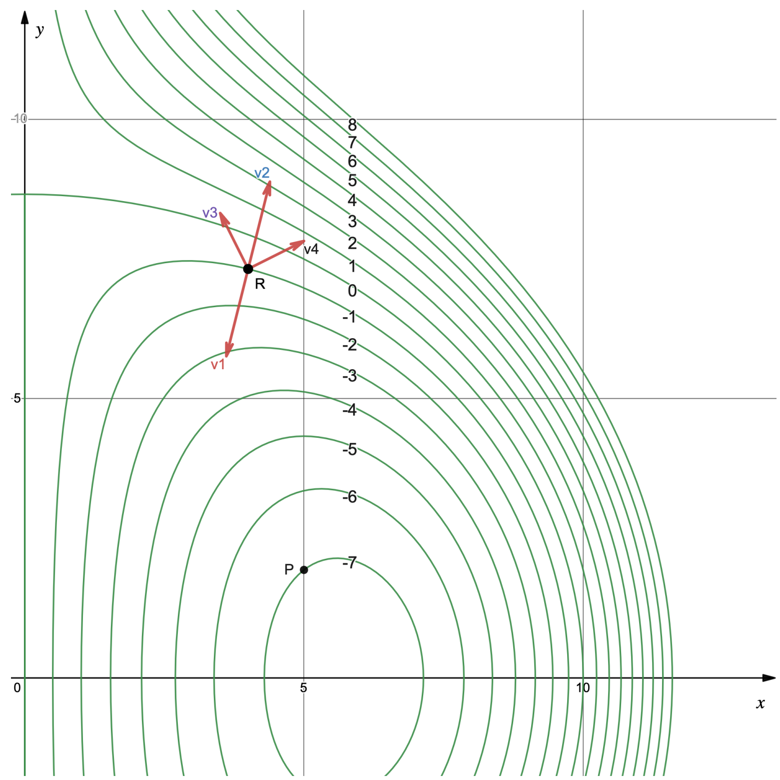
(v)



(f) A portion of the graph is reproduced below with four additional vectors $v1$, $v2$, $v3$, and $v4$ drawn on it. Which of the four vectors is $\nabla f(R)$?

- ☐ $v1$
- ☒ $v2$
- ☐ $v3$
- ☐ $v4$

direction of steepest ascent



Problem 2 (6 + 6 + 6 points)**.** Consider the vector $\vec{v} = \langle 2, -6, 9\rangle$.

(a) Find a unit vector parallel to $\vec{v}$.

$|\vec{v}| = \sqrt{2^2+(-6)^2+9^2} = 11$

Two solutions: $\frac{1}{11}\vec{v} = \langle \frac{2}{11}, \frac{-6}{11}, \frac{9}{11} \rangle$

& $-\frac{1}{11}\vec{v}$.

(b) Give an example of a vector $\vec{u} \neq \vec{v}$ that makes an obtuse angle with $\vec{v}$. *Briefly* explain your answer.

E.g. $\langle 0,1,0\rangle$ bc.

$\langle 0,1,0\rangle \cdot \langle 2,-6,9\rangle = -6 < 0$.

(c) Give an equation for a plane to which $\vec{v}$ is a normal vector.

$2x - 6y + 9z = 0$.

Problem 3 (10 points). Parametrize the line through the point $(3, 4, 5)$ that is normal to the plane $2x - 7y + z = 8$.

Normal vector $\vec{v} = \langle 2, -7, 1 \rangle$.

$$\langle 3, 4, 5 \rangle + t\vec{v} = \langle 3+2t, 4-7t, 5+t \rangle.$$

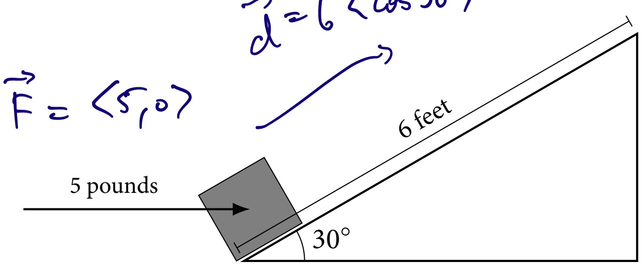
I.e. $x = 3+2t$
$y = 4-7t$
$z = 5+t$.

See Problem 9.3.8 in the textbook!

Problem 4 (8 points). A woman exerts a horizontal force of 5 pounds on a box as she pushes it all the way up a ramp that is 6 feet long and inclined at an angle of 30 degrees above the horizontal. Find, in foot-pounds, the work done on the box. Show your work.

$\vec{d} = 6\langle \cos 30°, \sin 30° \rangle$

$\vec{F} = \langle 5, 0 \rangle$



$$\text{Work} = \vec{F} \cdot \vec{d} \quad \left(= |\vec{F}|\,|\vec{d}| \cos 30° \right)$$

$$= \langle 5, 0 \rangle \cdot 6\langle \cos 30°, \sin 30° \rangle$$

$$= 5 \cdot 6 \cos 30° = 30\frac{\sqrt{3}}{2} = \boxed{15\sqrt{3} \text{ foot-pounds}}$$

Problem 5 (22 points). Consider the function

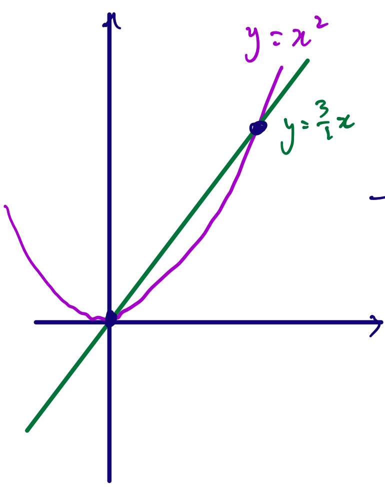
$$f(x, y) = x^3 - 3xy + y^2.$$

It has two critical points. Find them, and use the Second Derivative Test to attempt to classify each one. Show your work.

$f_x = 3x^2 - 3y$

$f_y = -3x + 2y$

Set $f_x = f_y = 0$:



$3x^2 - 3y = 0 \rightsquigarrow y = x^2$

$-3x + 2y = 0 \rightsquigarrow y = \frac{3}{2}x$

$\rightsquigarrow x^2 - \frac{3}{2}x = 0$

$\rightsquigarrow x\left(x - \frac{3}{2}\right) = 0$

$\rightsquigarrow x = 0$ (and $y = 0^2 = 0$)

or $x = \frac{3}{2}$ (and $y = \left(\frac{3}{2}\right)^2 = 9/4$).

$f_{xx} = 6x$

$f_{yy} = 2$

$f_{xy} = f_{yx} = -3$

$D = 6x \cdot 2 - (-3)^2$

$= 12x - 9.$

$D(0,0) = -9 < 0$

$D\left(\frac{3}{2}, \frac{9}{4}\right) = 18 - 9 = 9 > 0$

& $f_{yy}\left(\frac{3}{2}, \frac{9}{4}\right) = 2 > 0$

critical point #1: $(0, 0)$
- ☐ local min
- ☐ local max
- ☒ saddle point
- ☐ inconclusive

critical point #2: $\left(\frac{3}{2}, \frac{9}{4}\right)$
- ☒ local min
- ☐ local max
- ☐ saddle point
- ☐ inconclusive

See Problem 10.5.7 in the textbook!

Problem 6 (12 points). You are given the following information about a function $W(s,t) = F(u(s,t), v(s,t))$.

$$\begin{array}{lll} u(1,0) = 1 & u_s(1,0) = 3 & u_t(1,0) = 6 \\ v(1,0) = 3 & v_s(1,0) = 5 & v_t(1,0) = -5 \\ F_u(1,3) = 4 & F_v(1,3) = 9 & \end{array}$$

(a) Write down a Chain Rule for computing $W_s(1,0)$. (Do not compute it yet.)

$$\frac{\partial W}{\partial s}(1,0) = \frac{\partial F}{\partial u}\big(u(1,0), v(1,0)\big)\frac{\partial u}{\partial s}(1,0) + \frac{\partial F}{\partial v}\big(u(1,0), v(1,0)\big)\frac{\partial v}{\partial s}(1,0)$$

(b) Now compute $W_s(1,0)$. Show your work.

$$\frac{\partial F}{\partial u}\big(u(1,0), v(1,0)\big) = \frac{\partial F}{\partial u}(1,3) = 4$$

$$\frac{\partial F}{\partial v}\big(u(1,0), v(1,0)\big) = \frac{\partial F}{\partial v}(1,3) = 9.$$

$$u_s(1,0) = 3 \ ; \ v_s(1,0) = 5.$$

$$\text{Answer} = 4 \cdot 3 + 9 \cdot 5 = \boxed{57}$$

See Problem 10.4.9 in the textbook!

Problem 7 (6 + 6 + 6 points)**.** An unevenly heated metal plate has temperature $T(x, y)$ in degrees Celsius at a point (x, y). Suppose that $T(2, 1) = 145$, $T_x(2, 1) = 6$, and $T_y(2, 1) = -8$.

(a) Use the linearization to estimate the temperature at the point $(2.04, 0.95)$.

$$L(x,y) = 145 + T_x(2,1)(x-2) + T_y(2,1)(y-1)$$

$$= 145 + 6(x-2) - 8(y-1)$$

$$L(2.04, 0.95) = 145 + 6(0.04) - 8(-0.05) = \boxed{145.64^\circ}$$

(b) Find (exactly) the instantaneous rate of change of T at $(2, 1)$ in the direction $\left\langle \frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle$. (unit vector)

$$T_{\langle \frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \rangle}(2,1) = \nabla T(2,1) \cdot \left\langle \frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle$$

$$= \langle 6, -8 \rangle \cdot \left\langle \frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle$$

$$= \frac{-14}{\sqrt{2}} \ {}^\circ C/\text{unit}.$$

(c) At $(2, 1)$, in which direction is the temperature increasing most rapidly? Give your answer as a unit vector.

$$\frac{\nabla T}{|\nabla T|} = \frac{\langle 6, -8 \rangle}{|\langle 6, -8 \rangle|} = \boxed{\left\langle \frac{3}{5}, \frac{-4}{5} \right\rangle}$$