MATH 208, EXAM 3

SECTION 150

Name: Solutions NUID: ______

Instructions.

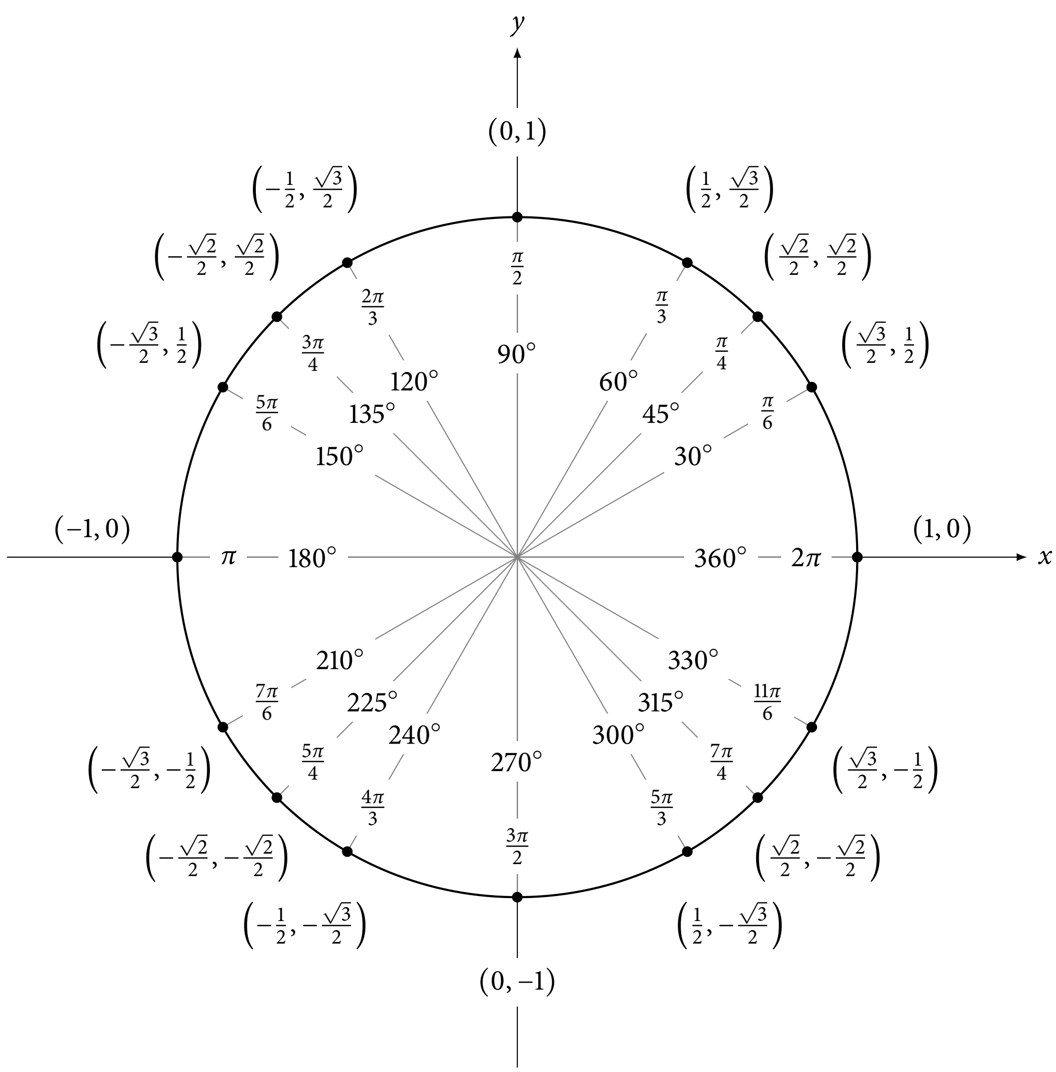
- You should have 8 pages on which 6 problems are printed.
- You have 50 minutes: the exam will begin on the half-hour and end promptly, 50 minutes later.
- Show all work unless otherwise specified. What you write on the page must convince me that you understand the problem and its solution.
- Read each problem carefully.
- You do not need to simplify your answers, unless the instructions for a problem indicate otherwise.
- You are not allowed a calculator, notes, textbooks, or access to any electronic devices.
- Don't panic. Good luck!

Date: Fall 2022.

Here are some things you might find useful.

$$x = \rho \sin\phi \cos\theta$$
$$y = \rho \sin\phi \sin\theta$$
$$z = \rho \cos\phi$$

$$dV = \rho^2 \sin\phi \, d\rho d\phi d\theta$$



Problem 1 (2 + 8 points). Consider the spiral curve parametrized by

$$\vec{r}(t) = \langle 3t, 4\sin t, 4\cos t\rangle \quad \text{for} \quad -1 \le t \le 3.$$

(a) Which of the following correctly describes the line tangent to this curve at the point $(0, 0, 4)$? (There is exactly one correct answer.)

- [ ] $3x + 4y + z = 0$ ← not even a line!
- [ ] $x = 3t$, $y = 4\sin t$, $z = 4\cos t$
- [x] $x = 3t$, $y = 4t$, $z = 4$
- [ ] $\vec{L}(t) = \langle 0, 0, 0\rangle + t\,\langle 0, 0, 4\rangle$

$\vec{v}(t) = \langle 3, 4\cos t, -4\sin t\rangle$

$\vec{v}(0) = \langle 3, 4, 0\rangle$.

$\langle 0,0,4\rangle + t\langle 3,4,0\rangle$

$= \langle 3t, 4t, 4\rangle$

(b) Find the arclength of the spiral curve.

Integrate speed:

$|\vec{v}(t)| = \sqrt{3^2 + (4\cos t)^2 + (-4\sin t)^2}$

$= \sqrt{9+16} = 5.$

$\int_{t=-1}^{3} 5\,dt = \boxed{20}$

Initials:

Math 208, Exam 3

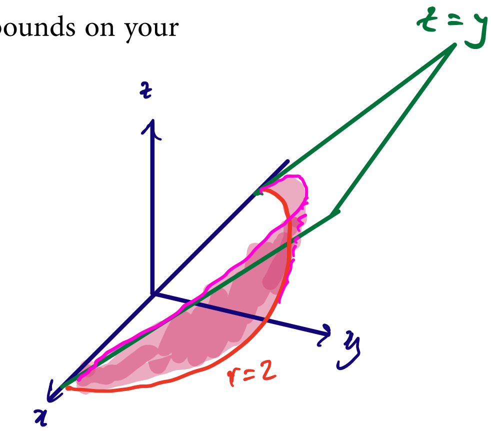
Problem 2 (8 + 12 points)**.** Let S be the portion of the cylinder $x^2 + y^2 = 4$ lying above the xy-plane and below the plane $z = y$. $= r\sin\theta$ $r = 2$

(a) Give a parametrization of the surface S. Be sure to give bounds on your parameters.

$$\vec{r}(\theta, z) = \langle 2\cos\theta, 2\sin\theta, z \rangle$$

$$0 \le \theta \le \pi$$

$$0 \le z \le 2\sin\theta$$



(b) Express the surface area of S as a double integral. **Do not evaluate your integral.**

$$\vec{r}_\theta \times \vec{r}_z = \langle -2\sin\theta, 2\cos\theta, 0 \rangle \times \langle 0, 0, 1 \rangle$$

$$= \langle 2\cos\theta, 2\sin\theta, 0 \rangle$$

"Speed II" $= |\vec{r}_\theta \times \vec{r}_z| = \sqrt{(2\cos\theta)^2 + (2\sin\theta)^2} = 2$

$$\boxed{\int_{\theta=0}^{\pi} \int_{z=0}^{2\sin\theta} 2 \, dz \, d\theta}$$

Problem 3 (15 + 5 points). Consider the following vector field in three dimensions.

$$\vec{F} = \langle e^{-2y}, 3z - 2xe^{-2y}, 3y \rangle$$

(a) Clearly state what it means for g to be a potential function for $\vec{F}$, and then find such a potential function g. (Show your work.)

Means $\nabla g = \vec{F}$.

$g_x = e^{-2y} \Rightarrow g(x,y,z) = xe^{-2y} + C(y,z)$.

So $3z - 2xe^{-2y} = g_y = -2xe^{-2y} + C_y$

So $C_y = 3z \Rightarrow C(y,z) = 3yz + D(z)$.

So $g(x,y,z) = xe^{-2y} + 3yz + D(z)$.

$3y = g_z = 3y + D'(z) \Rightarrow D'(z) = 0$.

$$\boxed{g(x,y,z) = xe^{-2y} + 3yz}$$

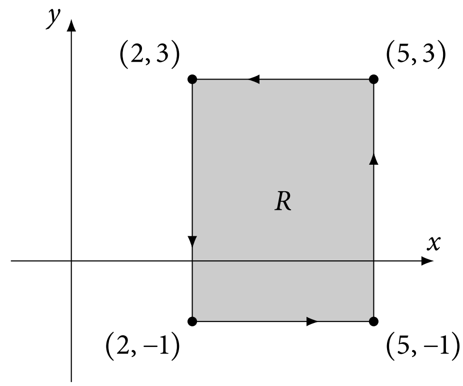
(b) Compute the line integral $\int_C \vec{F} \cdot d\vec{r}$ over any path C from $(1, 0, \pi)$ to $(0, 1, 2)$. Briefly explain why your answer depends only on the endpoints of C and not on its behavior between the endpoints.

By the Fundamental Thm of Calculus for Line Integrals,

$$\int_C \vec{F} \cdot d\vec{r} = \int_C \nabla g \cdot d\vec{r} = g(0,1,2) - g(1,0,\pi)$$

$$= 6 - 1 = \boxed{5}$$

Problem 4 (10 points)**.** Consider the rectangle R in the plane with vertices $(2,-1)$, $(5,-1)$, $(5,3)$, and $(2,3)$ and the vector field $\vec{F} = \left\langle 3y - e^{x^2}, e^{x^2+y^2} \right\rangle$. Use Green's Theorem to express the circulation of $\vec{F}$ around the boundary of R (oriented counter-clockwise) as a double integral. **Do not evaluate your integral.**



$$\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = 2xe^{x^2+y^2} - 3$$

$$\oint \vec{F} \cdot d\vec{r} = \iint \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \, dA$$

$$= \boxed{\int_{x=2}^{5} \int_{y=-1}^{3} \left(2xe^{x^2+y^2} - 3\right) dy\, dx}$$

Math 208, Exam 3 Initials:

Problem 5 (20 points). Compute the line integral $\int_C \vec{F}\cdot d\vec{r}$, where $\vec{F} = \langle -2e^y, -xz, xy\rangle$ and C is the straight-line path from $(1,0,0)$ to $(0,2,3)$.

Parametrize: $\vec{r}(t) = \langle 1,0,0\rangle + t\langle -1,2,3\rangle$

$$= \langle 1-t, 2t, 3t\rangle, \qquad 0 \le t \le 1.$$

$$\vec{r}\,'(t) = \langle -1, 2, 3\rangle.$$

$$\vec{F}(\vec{r}(t)) \cdot r'(t) = \langle 2e^{2t}, -(1-t)\cdot 3t, (1-t)\,2t\rangle \cdot \langle -1,2,3\rangle$$

$$= 2e^{2t} - 2\cancel{(1-t)3t} + 3\cancel{(1-t)2t}$$

$$= 2e^{2t}$$

$$\int_C \vec{F}\cdot d\vec{r} = \int_{t=0}^{1} 2e^{2t}\,dt = \left[e^{2t}\right]_0^1 = \boxed{e^2 - 1}$$

Problem 6 (20 points). Compute the flux of the vector field $\vec{F} = \langle -x, 2y, 3z \rangle$ over the portion of the plane $3x + 3z = 10$ that lies above the unit square $0 \leq x \leq 1$, $0 \leq y \leq 1$, oriented with upward-pointing normal vector.

Parametrize using $x = s$, $y = t$, so $z = \frac{1}{3}(10 - 3x) = \frac{1}{3}(10 - 3s)$.

$\vec{r}(s,t) = \langle s, t, \frac{1}{3}(10 - 3s) \rangle$, $0 \leq s \leq 1$, $0 \leq t \leq 1$.

$\vec{r}_s \times \vec{r}_t = \langle 1, 0, -1 \rangle \times \langle 0, 1, 0 \rangle$

$= \langle 1, 0, 1 \rangle$.

$\vec{F}(\vec{r}(s,t)) \cdot (\vec{r}_s \times \vec{r}_t) = \langle -s, 2t, 3(\frac{1}{3}(10 - 3s)) \rangle \cdot \langle 1, 0, 1 \rangle$

$= -s + 10 - 3s = 10 - 4s$.

$$\text{Flux} = \int_{s=0}^{1} \int_{t=0}^{1} 10 - 4s \, dt \, ds$$

$$= \int_{s=0}^{1} 10 - 4s \, ds = \left[10s - 2s^2\right]_0^1 = \boxed{8}$$