Math 208 Final Exam

Initials:

Name (write clearly) Solutions

NU ID number

Circle the name and time of your lecture:

Norwood 8:30 Norwood 9:30 Schafhauser 11:30 Rebarber 12:30 Burns 6:30 Yang 208H.

Instructions

- There are **15** questions on **17** pages (including this cover sheet and the formula sheet on the second page).
- No books, notes or calculator are allowed.
- Turn off all communication devices.
- **Show all your work** and explain your answers. Unsupported answers will receive **little credit**.
- If specified, use the method required by each problem. Alternate methods will not receive full credit.
- In multi-part problems, the parts might not be worth the same number of points.
- You have **2 hours** to complete the exam.

Good luck !

Question	Out of	Score
1	10	
2	10	
3	12	
4	12	
5	12	
6	12	
7	16	
8	12	
9	18	
10	16	
11	13	
12	12	
13	14	
14	17	
15	14	
TOTAL	200	

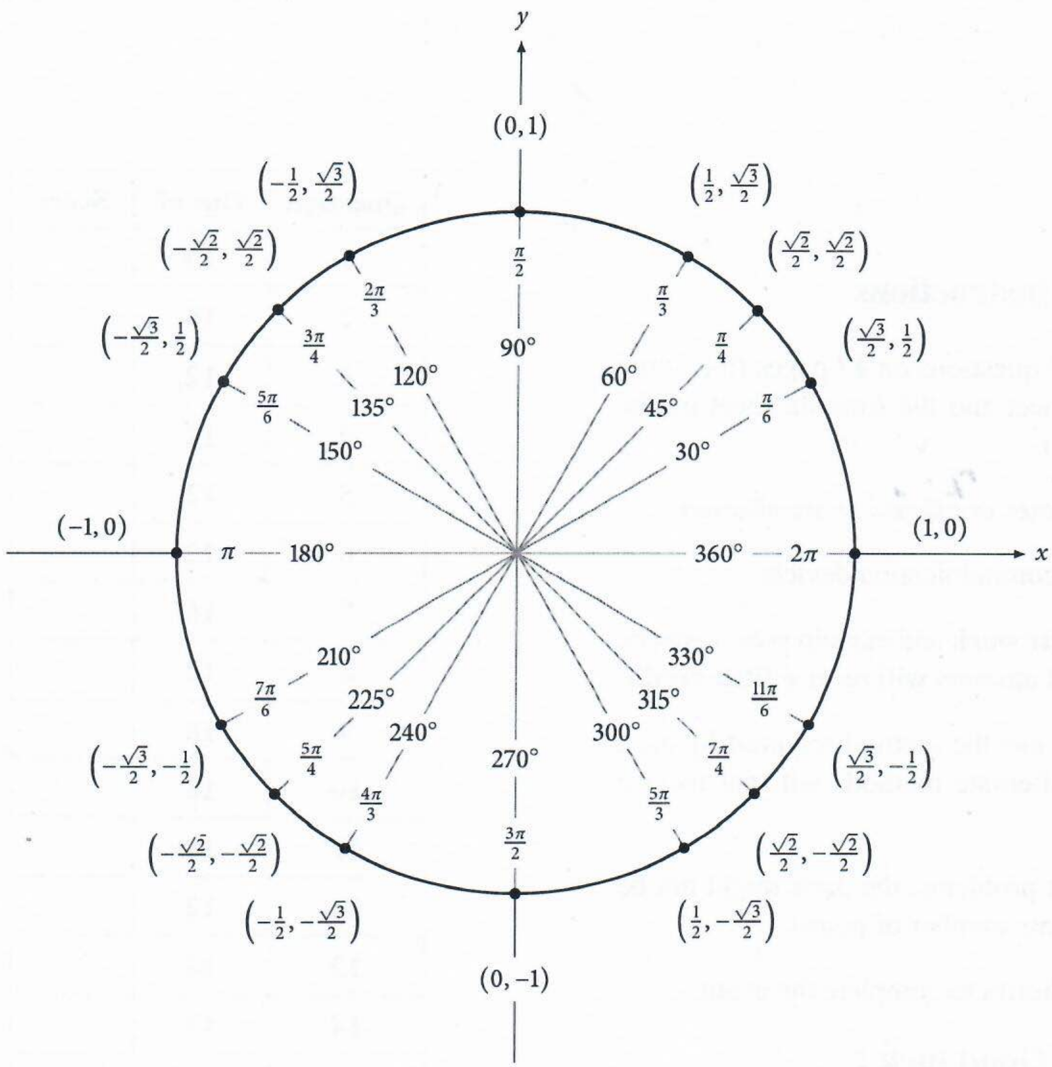
The formulas on the next page might or might not be useful:

$$x=\rho\sin\phi\cos\theta,\quad y=\rho\sin\phi\sin\theta,\quad z=\rho\cos\phi,\quad dV=\rho^2\sin\phi\,d\rho\,d\phi\,d\theta.$$

$$\int_C \vec{F}\cdot d\vec{r}=\pm\int\int \operatorname{curl}(\vec{F})\cdot(\vec{r}_s\times\vec{r}_t)\,ds\,dt$$

The flux of $\vec{F}$ over the boundary of W is equal to the integral of the divergence of $\vec{F}$ over W.

$$D=f_{xx}f_{yy}-f_{xy}^2.$$



1. (10 points)

(a) Find a normal vector to the plane containing the points $(0,1,3)$, $(-2,0,-1)$ and $(1,1,0)$.

$\vec{v}_1 = \langle -2, -1, -4 \rangle$ $\vec{v}_2 = \langle 1, 0, -3 \rangle$ (there are other choices of vectors)

$$\vec{n} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -2 & -1 & -4 \\ 1 & 0 & -3 \end{vmatrix} = \vec{i}(3) - \vec{j}(6+4) + k1 = \langle 3, -10, 1 \rangle$$

(b) Find an equation for the plane in part (a).

$$3(x-0) - 10(x-1) + 1(z-3) = 0$$

2. (10 points) Match each vector field formula with the corresponding graph.

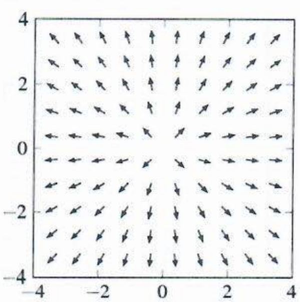
(i) $\langle 0, x\rangle$
- [ ] Graph A
- [ ] Graph B
- [x] Graph C
- [ ] Graph D
- [ ] Graph E

(ii) $\langle 0, y\rangle$
- [ ] Graph A
- [ ] Graph B
- [ ] Graph C
- [x] Graph D
- [ ] Graph E

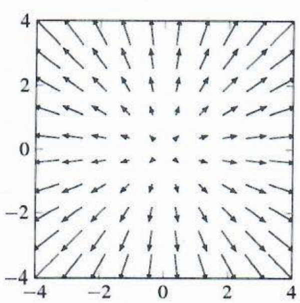
(iii) $\langle x, y\rangle$
- [ ] Graph A
- [x] Graph B
- [ ] Graph C
- [ ] Graph D
- [ ] Graph E

(iv) $\frac{1}{\sqrt{x^2+y^2}}\langle x, y\rangle$
- [x] Graph A
- [ ] Graph B
- [ ] Graph C
- [ ] Graph D
- [ ] Graph E

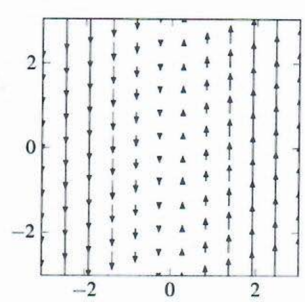
(v) $\langle |\sin(y)|, 0\rangle$
- [ ] Graph A
- [ ] Graph B
- [ ] Graph C
- [ ] Graph D
- [x] Graph E



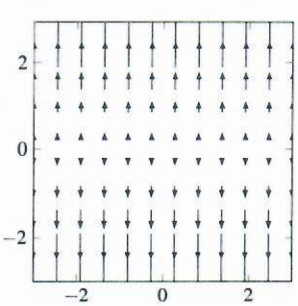
Graph A



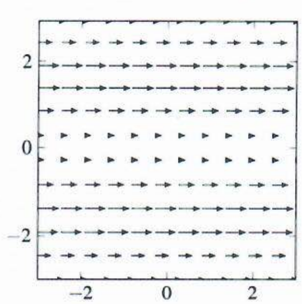
Graph B



Graph C



Graph D



Graph E

3. (12 points)

(a) Find a number α such that $2\vec{i}+\vec{j}+3\vec{k}$ is perpendicular to $\alpha\vec{i}+2\vec{j}-6\vec{k}$.

$2\alpha + 2 - 18 = 0$

$2\alpha = 16$

$\alpha = 8$

(b) Find the **cosine** of the angle between the two vectors $\vec{u} = -2\vec{i}+3\vec{j}+6\vec{k}$ and $\vec{v} = \vec{i}+2\vec{j}-\vec{k}$.

$$\cos\theta = \frac{-2+6-6}{\sqrt{4+9+36}\,\sqrt{1+4+1}} = \frac{-2}{7\sqrt{6}}$$

4. (12 points)

(a) Find the directional derivative of $f(x,y,z) = ze^y + x^2$ at $P = (5,0,-1)$ in the direction from the point P to the point $Q = (4,1,2)$.

$$\vec{u} = \frac{\langle -1, 1, 3 \rangle}{\sqrt{11}}$$

$$\vec{\nabla} f = \langle 2x, ze^y, e^y \rangle \qquad \vec{\nabla} f(5,0,-1) = \langle 10, -1, 1 \rangle$$

$$D_{\vec{u}} f(5,0,-1) = \langle 10, -1, 1 \rangle \cdot \frac{\langle -1, +1, +3 \rangle}{\sqrt{11}} = \frac{-8}{\sqrt{11}}$$

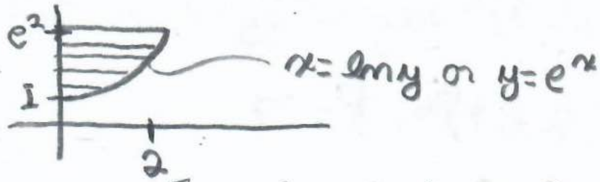
(b) What is the direction of maximum rate of change of f at P? Give the direction as a unit vector.

$$\vec{u} = \frac{\langle 10, -1, 1 \rangle}{\sqrt{102}}$$

5. (12 points)

(a) Draw the region of integration for

$$\int_1^{e^2}\int_0^{\ln y} dx\,dy.$$



$y = e^2 \Rightarrow x = \ln y = 2$

(b) Switch the order of integration for this integral. Do not evaluate.

$$\int_0^2 \int_{e^x}^{e^2} dy\,dx$$

6. (12 points)

(a) Find the local linearization of $f(x,y) = (xy^2+7)^{3/2}$ at the point $(2,1)$.

$f_x = \frac{3}{2}(xy^2+7)^{1/2}y^2 \Rightarrow f_x(2,1) = \frac{3}{2}(9)^{1/2} = \frac{9}{2}$

$f_y = \frac{3}{2}(xy^2+7)^{1/2}\,2xy \Rightarrow f_y(2,1) = \frac{3}{2}9^{1/2}(4) = \frac{9}{2}\cdot 4 = 18$

$f(2,1) = (2+7)^{3/2} = 3^3 = 27$

$f(x,y) \cong 27 + \frac{9}{2}(x-2) + 18(y-1)$

(b) For $f(x,y)$ in part (a), use the local linearization to approximate $f(2.05,.9)$. Leave your answer as a number, but there is no need to simplify it.

$f(2.05,.9) \cong 27 + \frac{9}{2}(.05) + 18(-.1)$

7. (16 points) (a) Find both **critical points** of $f(x,y) = 3xy - x^3 - y^3 + 3$.

$f_x = 3y - 3x^2 = 0 \Rightarrow y = x^2$

$f_y = 3x - 3y^2 = 0 \Rightarrow x = y^2 \Rightarrow x = x^4 \Rightarrow x = 0$ or $x^3 = 1$

$\Rightarrow \underbrace{x = 0}_{y=0}$ or $\underbrace{x = 1}_{y=1}$

$(0,0)$ and $(1,1)$

(b) Use the **Second Derivative Test** to classify each of the critical point(s) as a local maximum, local minimum, or saddle point.

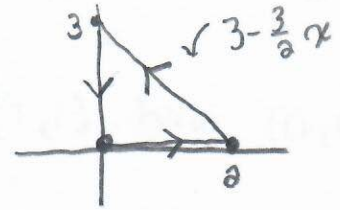
	$(0,0)$	$(1,1)$
$f_{xx} = -6x$	0	$-6 <$
$f_{yy} = -6y$	0	-6
$f_{xy} = 3$	3	3
D	-3	$36 - 9 > 0$
	saddle	local max

8. (12 points) Use **Green's Theorem** to evaluate the line integral

$$\int_C (2xy-3)dx+(x^2+x+y)dy,$$

where C is the close curve consisting of the line segment from $(0,0)$ to $(2,0)$, followed by the line segment from $(2,0)$ to $(0,3)$, followed by the line segment from $(0,3)$ back to $(0,0)$.

$$\iint_R \left[(2x+1)-2x\right] dA$$



$$= \iint_R 1\, dA = \frac{1}{2}(3)(2) = 3$$

Don't need to integrate, but if you do: $\int_0^2 \int_0^{3-\frac{3}{2}x} dy\, dx$

$$= \int_0^2 \left(3-\frac{3}{2}x\right) dx = 3x - \frac{3}{4}x^2 \Big|_0^2 = 6-3=3$$

9. (18 points) Let W be the part of the solid ball $x^2+y^2+z^2 \leq 16$ which is in the octant with

$$\{x \geq 0, y \leq 0, z \leq 0\}.$$

(a) Find an integral for the volume of W in spherical coordinates. Do not evaluate.

$\frac{3\pi}{2} < \theta < 2\pi$ $\qquad$ $z \leq 0 \Rightarrow \frac{\pi}{2} \leq \phi \leq \pi$

$$\int_{\frac{3\pi}{2}}^{2\pi} \int_{\pi/2}^{\pi} \int_{0}^{4} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

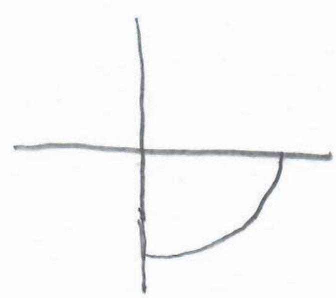
or $\int_{-\pi/2}^{0}$

(b) Find an integral for the volume of W in cartesian coordinates. Do not evaluate.

$$\int_{0}^{4} \int_{-\sqrt{16-x^2}}^{0} \int_{-\sqrt{16-x^2-y^2}}^{0} dz \, dy \, dx$$

or $$\int_{-4}^{0} \int_{0}^{\sqrt{16-y^2}} \int_{-\sqrt{16-x^2-y^2}}^{0} dz \, dx \, dy$$

(there are other correct ways)



10. (16 points) Let S be that part of the surface $z = xy + 5$ which is above the square

$$\{0 \le x \le 1,\ \ 1 \le y \le 2\}$$

in the (x,y) plane. Assume S is oriented with normals that have a positive $\vec{k}$ component. Find the flux of the vector field $\vec{F}(x,y,z) = \langle x, 0, xy+1\rangle$ through S.

$x = s$
$y = t$
$z = st+5$

$\vec{r}(s,t) = \langle s, t, st+5\rangle$

$\vec{r}_s = \langle 1, 0, t\rangle$

$\vec{r}_t = \langle 0, 1, s\rangle$

$\vec{r}_s \times \vec{r}_t = \vec{i}(-t) - \vec{j}(s) + \vec{k}(1) = \langle -t, -s, 1\rangle$

$\vec{r}_s \times \vec{r}_t = \langle -t, -s, 1\rangle$

$\int_1^2 \int_0^1 \vec{F} \cdot \langle -t,-s,1\rangle \, ds\, dt = \int_1^2 \int_0^1 \langle s, 0, ts+1\rangle \cdot \langle -t, -s, 1\rangle \, ds\, dt$

$= \int_1^2 \int_0^1 \underbrace{(-st + ts + 1)}_{1} \, ds\, dt = 1$

11. (13 points) Use **Lagrange multipliers** to find the maximum and minimum values of $f(x,y) = x+3y$ subject to the constraint $x^2+y^2 = 4000$.

$1 = \lambda 2x$
$3 = \lambda 2y$ $\quad 3 = \frac{y}{x} \Rightarrow y = 3x$
$x^2+y^2 = 100$

$(3x)^2 + x^2 = 4000 \Rightarrow y^2 = 400 \Rightarrow x = 20, -20$
$y = 60, (-60)$

$f(20,60) = 20+180 = 200$ max at $(20,60)$
$f(-20,-60) = -200$ min at $(-20,-60)$

12. (12 points) Let $\vec{F}(x,y,z) = -3y\vec{i} + x\vec{j}$ and let C be that part of the curve $y = x^2$ from $(-1,1)$ to $(1,1)$. Evaluate the line integral $\int_C \vec{F} \cdot d\vec{r}$.

$x = t$ $\quad t; -1 \to 1$

$y = t^2$

$$\int_{-1}^{1} -3t^2\,dt + t\,2t\,dt = \int_{-1}^{1} -t^2\,dt = -\frac{t^3}{3}\Big|_{-1}^{1} = -\frac{1}{3} - \left(+\frac{1}{3}\right) = -\frac{2}{3}$$

13. (14 points) Let R be the solid region bounded below by $z=x^2+y^2$ and above by $z=32-(x^2+y^2)$. Let S be the boundary surface of R. Use the **Divergence Theorem** to find the outward flux of the vector field $\vec{F}=(x^3+y^3)\vec{i}+(y^3+xz)\vec{j}+(y-x^2y)\vec{k}$ through the surface S. Just set up the integral.

$$\int_0^{2\pi}\int_0^{4}\int_{r^2}^{32-r^2}(3x^2+3y^2)\,dz\,r\,dr\,d\theta$$

$r^2 = 32-r^2$

$\Rightarrow r=\pm 4$

$$=\int_0^{2\pi}\int_0^{4}\int_{r^2}^{32-r^2}3r^3\,dz\,dr\,d\theta$$

14. (17 points) Let C be the curve $x^2+y^2=25$ in the plane $z=1$, oriented counterclockwise when viewed from above. Let $\vec{F}=\langle z,x,y\rangle$. Let S be the disk enclosed by C, that is, the surface defined by $x^2+y^2\leq 25$ in the plane $z=1$.

(a) Find parametric equations for S, with parameters s and t.

$x = s\cos t$ $\quad 0 \leq s \leq 5$

$y = s\sin t$ $\quad 0 \leq t \leq 2\pi$

$z = 1$

(b) Use **Stokes' Theorem** to find a flux integral which is equivalent to

$$\int_C \vec{F}\cdot d\vec{r}.$$

Write the integral completely in terms of s and t. Do not evaluate the integral.

$\vec{r}(s,t) = \langle s\cos t, s\sin t, 1\rangle$

$\vec{r}_s = \langle \cos t, \sin t, 0\rangle$

$\vec{r}_t = \langle -s\sin t, s\cos t\ 0\rangle$

$\vec{r}_s \times \vec{r}_t = \vec{i}(0) - \vec{j}(0) + \vec{k}\, s$

$\text{curl } \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial_x & \partial_y & \partial_z \\ z & x & y \end{vmatrix} = \vec{i}(1) - \vec{j}(-1) + \vec{k}\,1 = \langle 1,1,1\rangle$

$(\text{curl } \vec{F})\cdot(\vec{r}_s \times \vec{r}_t) = s$

$\int_0^{2\pi}\int_0^5 s\, ds\, dt$

15. (14 points) (a) Show that this vector field is conservative:

$$\vec{F}(x,y) = \left(\frac{2x}{y} + 2x\right)\vec{i} + \frac{-x^2}{y^2}\vec{j}.$$

$N_x = \frac{-2x}{y^2}$, $M_y = \frac{-2x}{y^2}$

(b) Find a potential function for the vector field in part (a).

$f_x = \frac{2x}{y} + 2x \Rightarrow f = \frac{x^2}{y} + x^2 + \underset{\text{no } x}{\underline{\qquad}}$

$f_y = \frac{-x^2}{y^2} \Rightarrow f = \frac{x^2}{y} + \underset{\text{no } y}{\underline{\quad x^2 \quad}}$

$f = \frac{x^2}{y} + x^2$

(c) Find the work done by the force field in part (a) in moving an object from $(2,5)$ along a curve C to $(2,1)$.

$f(2,1) - f(2,5) = \left(\frac{4}{1} + 4\right) - \left(\frac{4}{5} + 4\right) = 4 - \frac{4}{5} = \frac{16}{5}$