Last time:

Vectors ⤳ directions w/ quantities:

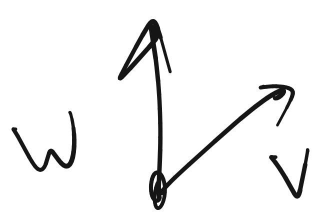
$$\vec{v} = \langle a, b, c \rangle = a\hat{i} + b\hat{j} + c\hat{k}$$

lie on same line!

two vectors $\vec{v}, \vec{w}$ are <u>parallel</u> if there is a

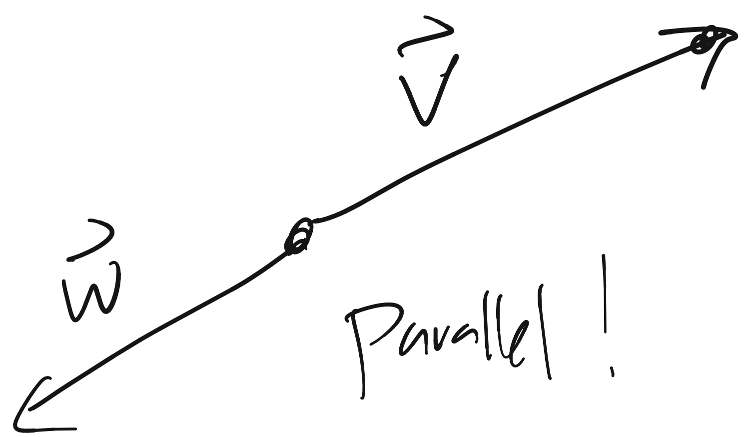
$\lambda \neq 0$ st $\vec{v} = \lambda \vec{w}$.

↑
scalar



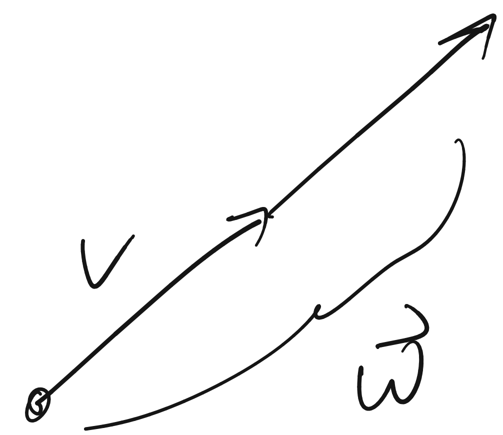
not parallel.

$\vec{v} \nparallel \vec{w}$



Parallel!

$\vec{v} \parallel \vec{w}$



Parallel

The **norm** or **magnitude** of $\vec{v} = \langle v_1, \dots, v_n \rangle$ is

$$\|\vec{v}\| = \sqrt{v_1^2 + v_2^2 + \dots + v_n^2}$$

Each "slot" is a **component** of $\vec{v}$

Dot product: returns a **scalar**

Setup: $\vec{v} = \langle v_1, \dots, v_n \rangle$

$\vec{w} = \langle w_1, \dots, w_n \rangle$

NO:

$$\cancel{\vec{v} \cdot \vec{w} \neq \langle v_1 w_1, v_2 w_2, \dots, v_n w_n \rangle}$$

Def'n: $\vec{v} \cdot \vec{w} = v_1 w_1 + v_2 w_2 + \dots + v_n w_n$

$$= \sum_{i=1}^{n} v_i w_i$$

Ex $\langle 1, 2, 3 \rangle \cdot \langle 3, -4, 1 \rangle$

$= 1\cdot 3 + 2(-4) + 3\cdot 1 = 3 - 8 + 3 = \boxed{-2}$

Ex Work

$\text{Work} = \overrightarrow{\text{Force}} \cdot \overrightarrow{\text{displacement}}$

Ex find work done on a block using $\vec{F} = \langle 10, 30 \rangle$ N

$\vec{d} = \langle -3, 2 \rangle$ m

Start point
End point

$\vec{d} = \text{final} - \text{initial}$

$$W = \vec{F}\cdot\vec{d} = \langle 10, 30\rangle \cdot \langle -3, 2\rangle$$

$$= 10\cdot(-3) + 30(2)$$

$$= -30 + 60 = 30 \text{ N}\cdot\text{m} \text{ or } 30 \text{ J}$$

Properties of dot product:

1. $\vec{u}\cdot\vec{v} = \vec{v}\cdot\vec{u}$

2. $(\vec{u}+\vec{v})\cdot\vec{w} = \vec{u}\cdot\vec{w} + \vec{v}\cdot\vec{w}$

3. if λ scalar, $(\lambda\vec{u})\cdot\vec{v} = \lambda(\vec{u}\cdot\vec{v})$

Theorem: $\vec{u} \cdot \vec{u} = \|\vec{u}\|^2$

proof: $\vec{u} = \langle u_1, \dots, u_n \rangle$

$$\vec{u} \cdot \vec{u} = u_1 u_1 + u_2 u_2 + \cdots + u_n u_n$$

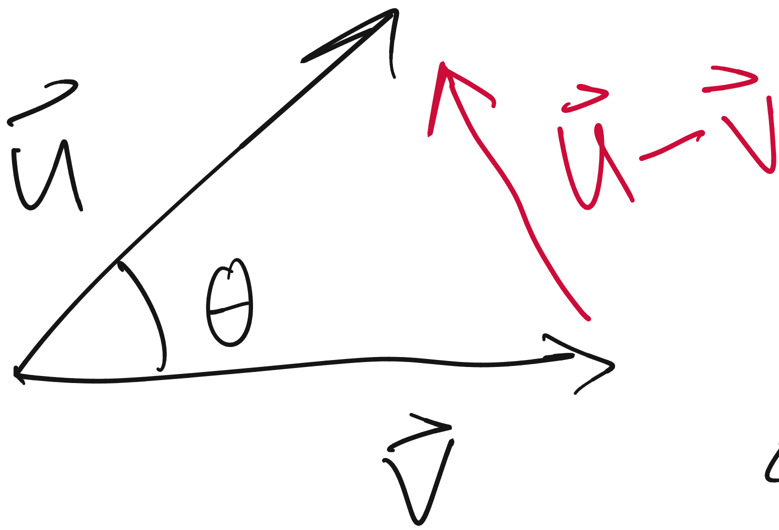
$$= u_1^2 + u_2^2 + \cdots + u_n^2$$

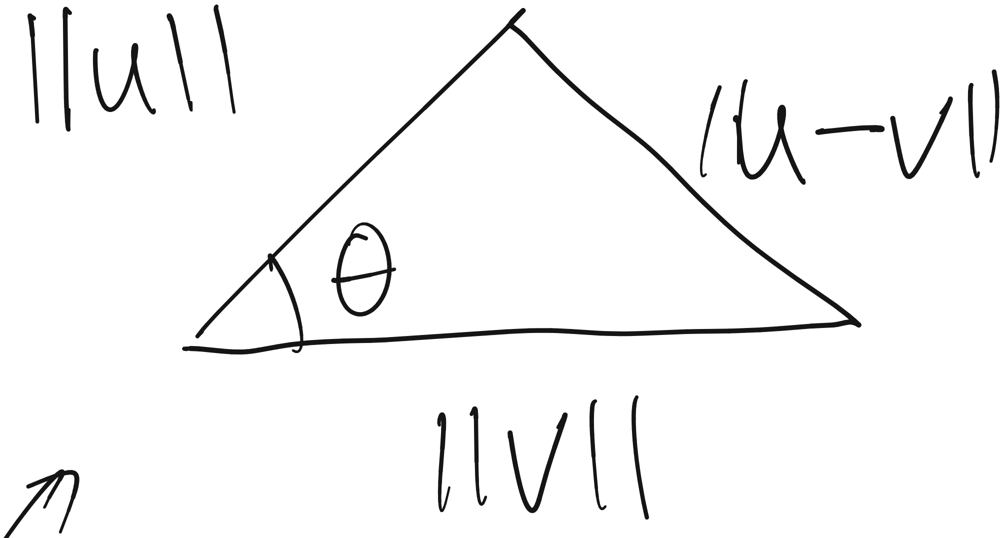
$$= \left(\sqrt{u_1^2 + u_2^2 + \cdots + u_n^2}\right)^2$$

$$= \|\vec{u}\|^2 \qquad \square$$

Halmos Tombstone

How find angle between two vectors?





Law of cosines tells us: $c^2 = a^2 + b^2 - 2ab\cos\theta$

$$\|\vec u - \vec v\|^2 = \|\vec u\|^2 + \|\vec v\|^2 - 2\|\vec u\|\,\|\vec v\|\cos\theta$$

$$\|\vec w\|^2 = \vec w \cdot \vec w$$

$$\|\vec{u}-\vec{v}\|^2 = (\vec{u}-\vec{v})\cdot(\vec{u}-\vec{v})$$

$$= \vec{u}\cdot\vec{u} - 2\vec{u}\cdot\vec{v} + \vec{v}\cdot\vec{v}$$

$$= \|\vec{u}\|^2 - 2\vec{u}\cdot\vec{v} + \|\vec{v}\|^2$$

$$\Rightarrow \cancel{\|u\|^2} + \cancel{\|v\|^2} - 2\|u\|\,\|v\| \cos\theta = \cancel{\|u\|^2} + \cancel{\|v\|^2} - 2\vec{u}\cdot\vec{v}$$

Scalar

$$\boxed{\vec{u}\cdot\vec{v} = \|\vec{u}\|\cdot\|\vec{v}\| \cos\theta}$$ ← Adjunction formula

$\vec{u} = \langle 1, 0 \rangle$ $\quad \|\vec{u}\| = \sqrt{1^2+0^2} = \sqrt{1} = 1$

$\vec{v} = \langle 1, 1 \rangle$ $\quad \|\vec{v}\| = \sqrt{1^2+1^2} = \sqrt{2}$

find θ between $\vec{u}, \vec{v}$. $\quad \vec{u} \cdot \vec{v} = 1 \cdot 1 + 0 \cdot 1 = 1$

$$1 = \underline{1 \cdot \sqrt{2}} \cos \theta$$

$$\frac{1}{\sqrt{2}} = \cos \theta \rightsquigarrow \theta = \cos^{-1}\left(\frac{1}{\sqrt{2}}\right) = \frac{\pi}{4} \text{ or } 45^\circ$$

<u>Result:</u> $\vec{v}, \vec{w}$ are non-zero vectors

then $\theta = \cos^{-1}\left(\dfrac{\vec{v}\cdot\vec{w}}{\|\vec{v}\|\,\|\vec{w}\|}\right)$

$\uparrow$

acos(~)

① find $\|\vec{v}\|, \|\vec{w}\|$

② find $\vec{v}\cdot\vec{w}$

③ Compute $\dfrac{\vec{v}\cdot\vec{w}}{\|\vec{v}\|\cdot\|\vec{w}\|}$

④ take $\cos^{-1}(\quad)$

Q what is $\cos(\pm 90°) = 0$

Thm: two non-zero vectors $\vec{u}, \vec{v}$ are perpendicular if & only if $\vec{u} \cdot \vec{v} = 0$.

ie $\vec{u}$ & $\vec{v}$ form a $90°$ angle only when $\vec{u} \cdot \vec{v} = 0$

Proof: if $\vec{u} \perp \vec{v}$ then $\theta = \pm 90°$ so $\cos\theta = 0$

So $\vec{u} \cdot \vec{v} = \|u\| \cdot \|v\| \cos\theta = 0$

on the other hand, if $\vec{u} \cdot \vec{v} = 0$ and since $\|u\|, \|v\| \neq 0$

we must have $\cos\theta = 0 \Rightarrow \theta = \pm 90°$, so

$\vec{u} \perp \vec{v}$. $\square$

Determine if two vectors $\vec{u}, \vec{v}$ are parallel, perp., or neither.

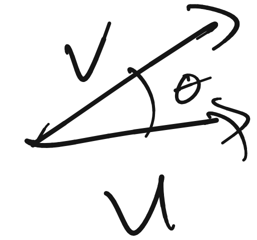
① Check perp. first by doing dot product.

↳ if $\vec{u} \cdot \vec{v} = 0$, $\vec{u} \perp \vec{v}$ done!

② Check / look for a $\lambda \neq 0$ st $\vec{u} = \lambda \vec{v}$

③ If no such λ exists, <u>Neither</u>

Rule of thumb:



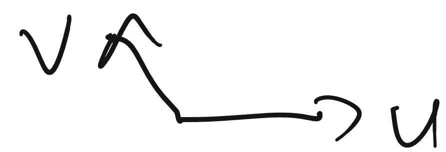
$-90° < \theta < 90°$

① if $\vec{u} \cdot \vec{v} > 0$, θ is acute angle

ie u, v point "in similar directions"

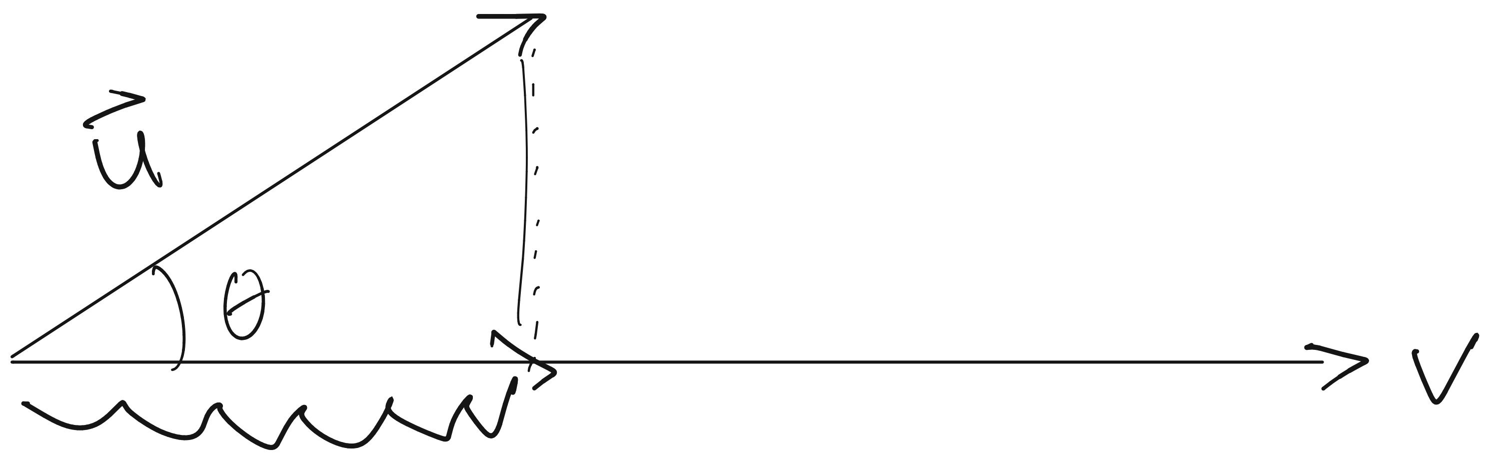
② if $\vec{u} \cdot \vec{v} = 0$ $\theta = \pm 90°$ $\vec{u} \perp \vec{v}$

③ if $\vec{u} \cdot \vec{v} < 0$, θ is obtuse, u, v point in "opposite directions"



$|\theta| > 90°$

Projection:



$\text{Proj}_{\vec{v}}(\vec{u})$

$$\text{Proj}_{\vec{v}}(\vec{u}) = \frac{\vec{u}\cdot\vec{v}}{\vec{v}\cdot\vec{v}}\,\vec{v}$$

"proj. of $\vec{u}$ in v direction"

or

Proj of $\vec{u}$ onto $\vec{v}$

Ex $u = \langle 2, 3, 5 \rangle$

$v = \langle 1, 1, 0 \rangle$

$u \cdot v = 5$

$v \cdot v = 2$

$\text{Proj}_v(u) = \frac{5}{2}\langle 1, 1, 0 \rangle$

$= \langle \frac{5}{2}, \frac{5}{2}, 0 \rangle$