Yesterday $\rightarrow$ $\boxed{\text{dot product}}$ "Scalar product" "inner product"

$\vec{u} = \langle u_1, \dots, u_n \rangle$

$\vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2 + \dots + u_n v_n$

$\vec{v} = \langle v_1, \dots, v_n \rangle$

$\vec{u} \cdot \vec{v} = \|\vec{u}\| \cdot \|\vec{v}\| \cos(\theta)$

$\hookrightarrow$ $\vec{u} \cdot \vec{v} = 0$ then $\vec{u} \perp \vec{v}$

Motivating question: $\vec{u}, \vec{v}$ $\vec{u} \nparallel \vec{v}$ how do we find

a $\vec{w}$ st $\vec{w} \perp \vec{u}$, $\vec{w} \perp \vec{v}$?

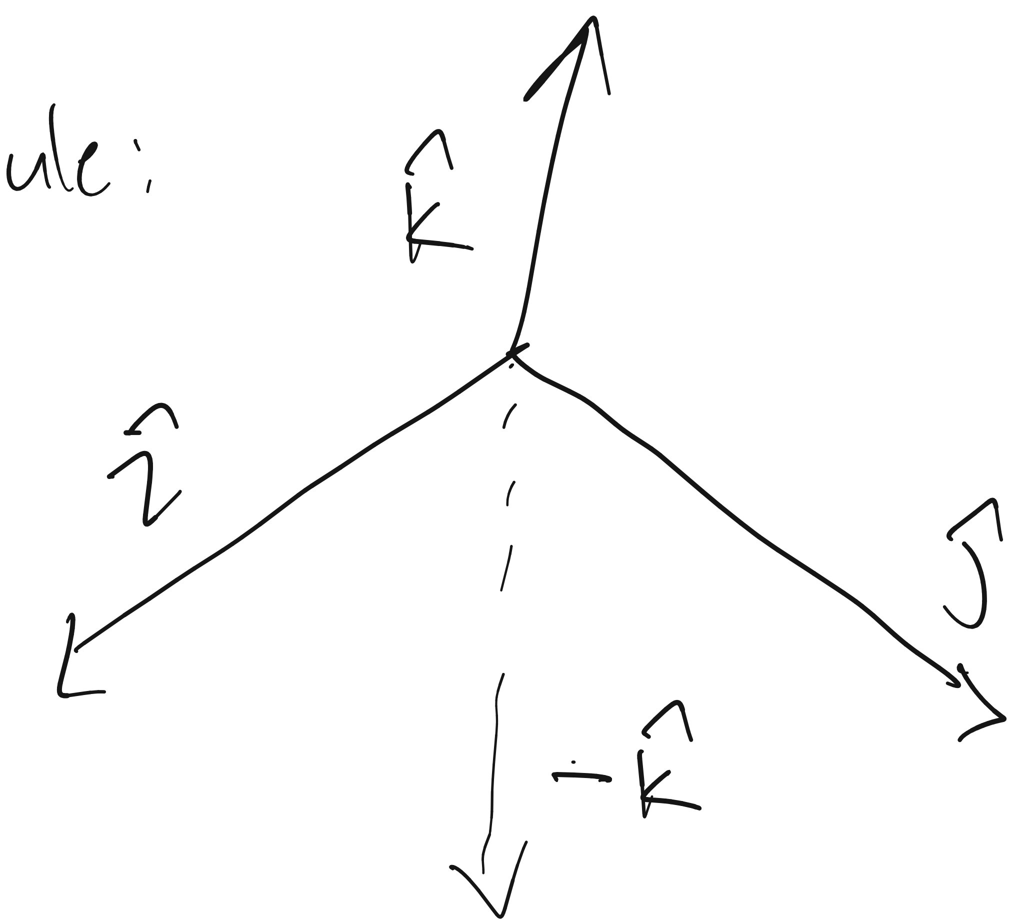
Recall: $\hat{\imath} = \langle 1, 0, 0 \rangle$

$\hat{\jmath} = \langle 0, 1, 0 \rangle$

$\hat{k} = \langle 0, 0, 1 \rangle$

$$\langle a, b, c \rangle = \underbrace{a\hat{\imath} + b\hat{\jmath} + c\hat{k}}$$

Right-hand rule:



Motivating Example: find a vector $\vec{w}$ that's perp to both

$\hat{\imath}$ and $\hat{\jmath}$ and satisfies RHR

$$(\hat{\imath}, \hat{\jmath}, \vec{w})$$

$\Rightarrow$ "Natural" sol'n is $\vec{w} = \pm \hat{k}$

What we just saw is that

$$\hat{\imath} \times \hat{\jmath} = \hat{k}$$

$\uparrow$

cross product of two vectors.

Def'n the cross product of two 3-D vectors

$\vec{u}, \vec{v}$ is a vector $\vec{w} = \vec{u} \times \vec{v}$ st

$w \perp u$, $w \perp v$ and satisfies the equation

$$\vec{u} \times \vec{v} = \|\vec{u}\| \|\vec{v}\| \sin\theta \, \hat{n}$$

finding $\hat{n}$ is hard!

$\hat{n}$ obeys RHR and is a unit vector.

Def'| (Algebraic definition)

$$\hat{i} \times \hat{j} = \hat{k}$$

$$\hat{j} \times \hat{k} = \hat{i}$$

$$\hat{k} \times \hat{i} = \hat{j}$$

$$\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$$

for any vectors $\vec{a}, \vec{b}$.

anti-commutativity.

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$u = \langle u_1, u_2, u_3 \rangle$

$v = \langle v_1, v_2, v_3 \rangle$

$$\vec{u} \times \vec{v} = (u_2 v_3 - u_3 v_2)\,\hat{i}$$
$$^* - (u_1 v_3 - u_3 v_1)\,\hat{j}$$
$$^* + (u_1 v_2 - u_2 v_1)\,\hat{k}$$

Ex $\langle -1, 0, 1 \rangle \times \langle 1, 0, 1 \rangle = \langle 0, 2, 0 \rangle$

$$= (0 \cdot 1 - 0 \cdot 1)\hat{i} - (-1 \cdot 1 - 1 \cdot 1)\hat{j}$$

$$+ (-1 \cdot 0 - 1 \cdot 0)\hat{k} = +2\hat{j}$$

Def'n 3

$\vec{u} = \langle a, b, c \rangle$

$\vec{v} = \langle d, e, f \rangle$

$$u \times v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a & b & c \\ d & e & f \end{vmatrix}$$

Ex $\vec{a} = \hat{i} + \hat{j} + 4\hat{k}$

$\vec{b} = 2\hat{i} - \hat{j} + 0\hat{k}$

$$\begin{array}{ccc} + & - & + \end{array}$$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 4 \\ 2 & -1 & 0 \end{vmatrix} = \hat{i}\left(1 \cdot 0 - (-1)4\right)$$

$$- \hat{j}\left(1 \cdot 0 - 2 \cdot 4\right)$$

$$+\hat{k}\left(1\cdot 1 - 1\cdot 2\right)$$

$$= \boxed{4\hat{i} + 8\hat{j} - 3\hat{k}} = \vec{c}$$

$$\vec{a}\cdot\vec{c} = 1\cdot 4 + 1\cdot 8 - 3\cdot 4 = 4+8-12 = 0$$

$\underline{Ex}$ $\langle 3,0,3\rangle \times \langle 4,1,2\rangle$

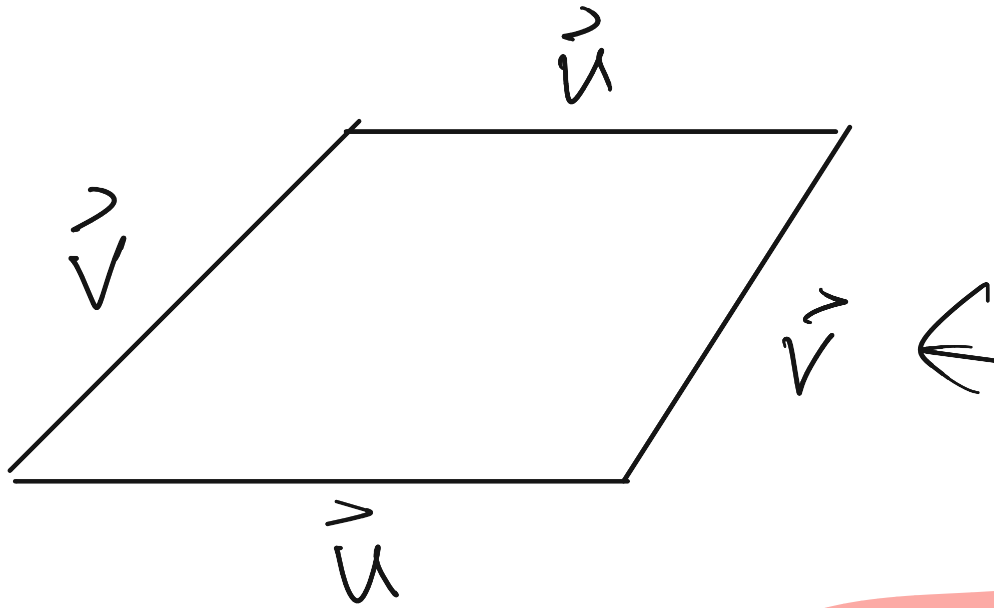
$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 0 & 3 \\ 4 & 1 & 2 \end{vmatrix} = \begin{aligned} &\hat{i}\,(0\cdot 2 - 1\cdot 3) \\ &-\hat{j}\,(3\cdot 2 - 4\cdot 3) \\ &+\hat{k}\,(3\cdot 1 - 0\cdot 4) \end{aligned}$$

$$= -3\hat{i} + 6\hat{j} + 3\hat{k}$$

$$= \langle -3, 6, 3\rangle.$$

Applications of Cross product:

(1)

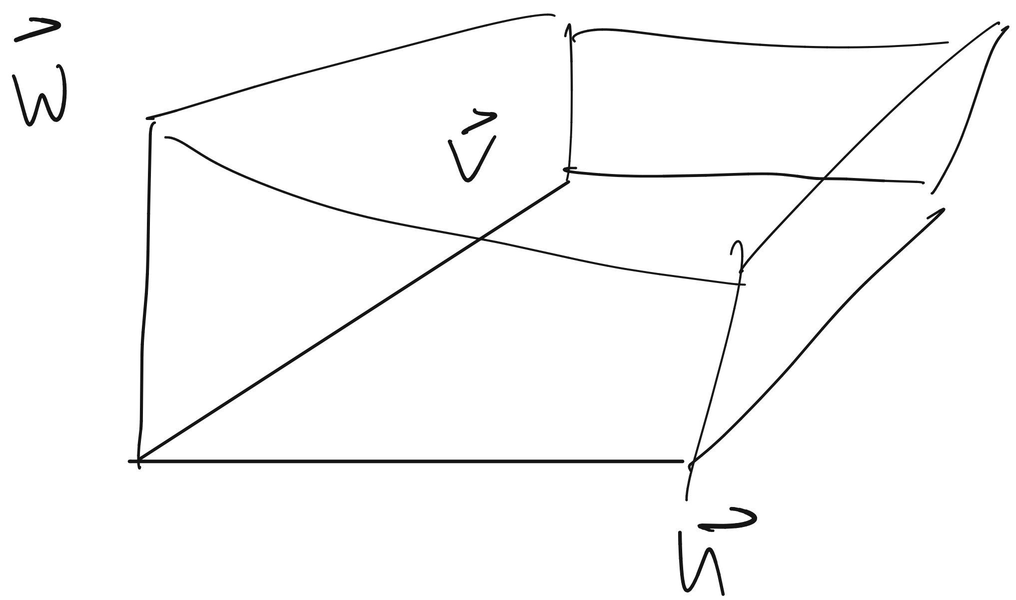


Cool fact: $\|\vec{u}\times\vec{v}\| = \text{Area(parallelogram)}$

Recall: $\vec{u}\times\vec{v} = \|u\|\cdot\|v\| \sin\theta\ \hat{n}$

$\|\vec{u}\times\vec{v}\| = \|u\|\cdot\|v\|\cdot|\sin\theta| = \text{Area}.$

② : Parallelepiped → 3D parallelogram.



triple Product

$$\text{Vol}(\text{Parallelepiped}) = \left| (\vec{u} \times \vec{v}) \cdot \vec{w} \right|$$

Ex $u = \hat{i}$

$v = \hat{j}$

$w = \hat{k}$

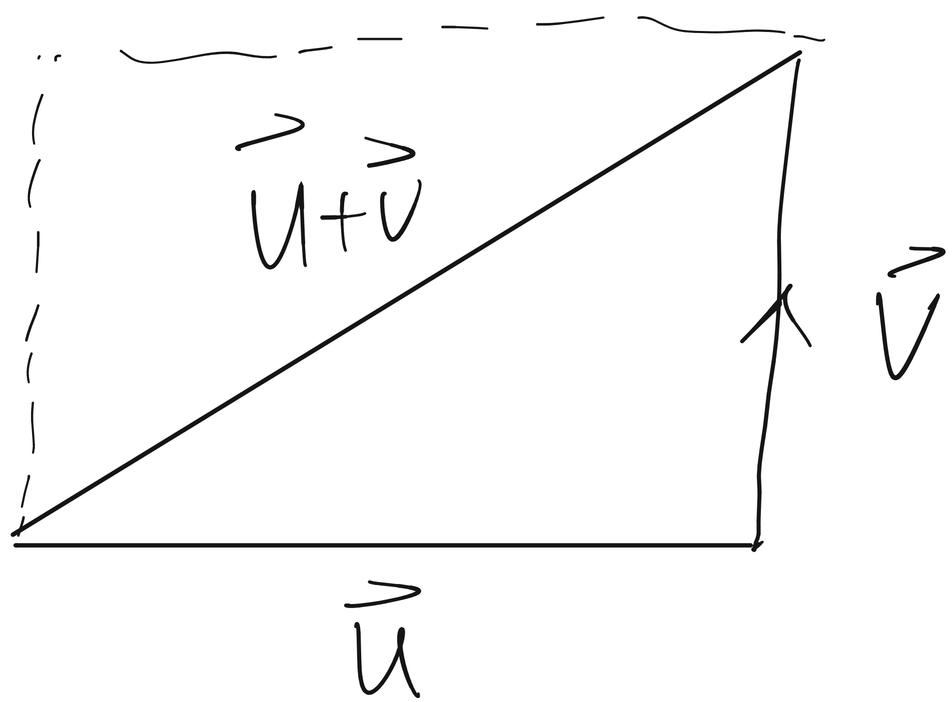
$$\vec{u} \times \vec{v} = \hat{k}$$

$$\text{Vol}(\ ) = |1| = 1$$

$$\hat{k} \cdot \vec{w} = \hat{k} \cdot \hat{k} = \langle 0\ 0\ 1 \rangle \cdot \langle 0\ 0\ 1 \rangle = 1$$

$$(u \times v) \cdot w = 1$$

③ triangle from vectors



$$\text{Area}(\Delta) = \frac{1}{2}\|\vec{u}\times\vec{v}\|$$

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④ Torque:

$$\vec{\tau} = \vec{F}\times\vec{d}$$

Properties of Cross product:

① : $\vec{u} \times \vec{v} = -(\vec{v} \times \vec{u})$

② : $(\vec{u}+\vec{v}) \times \vec{w} = \vec{u} \times \vec{w} + \vec{v} \times \vec{w}$

③ : $(\lambda \vec{u}) \times \vec{v} = \lambda (\vec{u} \times \vec{v})$

④ : $\vec{u} \times \vec{v} = \vec{0}$ if $\vec{u}, \vec{v}$ are parallel

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a & b & c \\ \lambda a & \lambda b & \lambda c \end{vmatrix} = 0\hat{i} + 0\hat{j} + 0\hat{k}$$