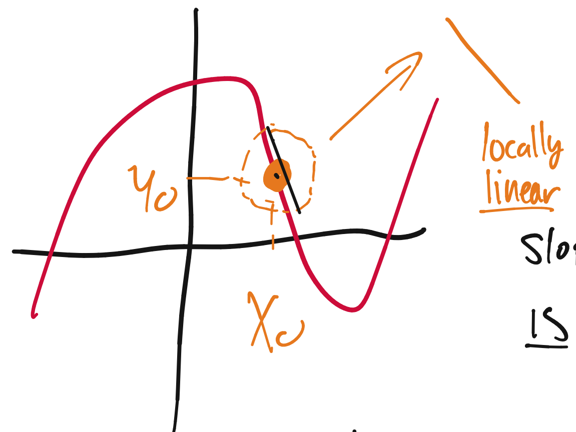
Today: §10.4 Tangent Plane & Differentials.

Tangent line:



$y = f(x)$

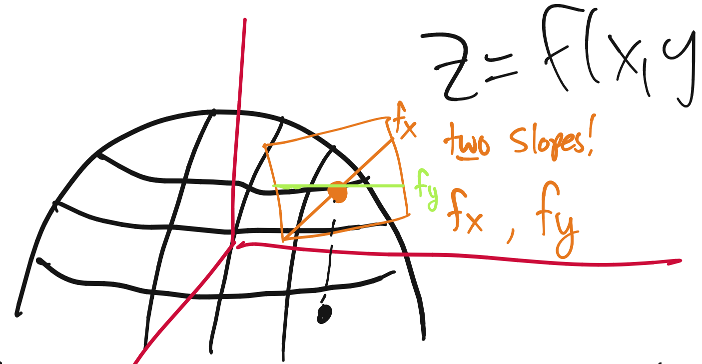
slope of tang. line is $f'(x_0)$

$y = f'(x_0)(x - x_0) + y_0$ ← Eqn for tangent line.

↑ $f(x_0)$

Similar idea holds in 3D!

Tangent Plane:



$z = f(x,y)$

Eqn: $z = f(x,y)$ and point (x_0, y_0) on x-y plane

z_0

Eqn is $z = f_x(x_0,y_0)\cdot(x - x_0) + f_y(x_0,y_0)\cdot(y - y_0) + f(x_0,y_0)$ ←

$\underline{Ex}$ $f(x,y) = x^2 y$ find tangent plane to graph of $f(x,y)$ @ point $(1,2)$.

$\underline{\text{Ingredients:}}$

$f(x_0, y_0) = f(1,2) = 1^2 \cdot 2 = 2$

$f_x = 2xy$ $f_x(1,2) = 2 \cdot 1 \cdot 2 = 4$

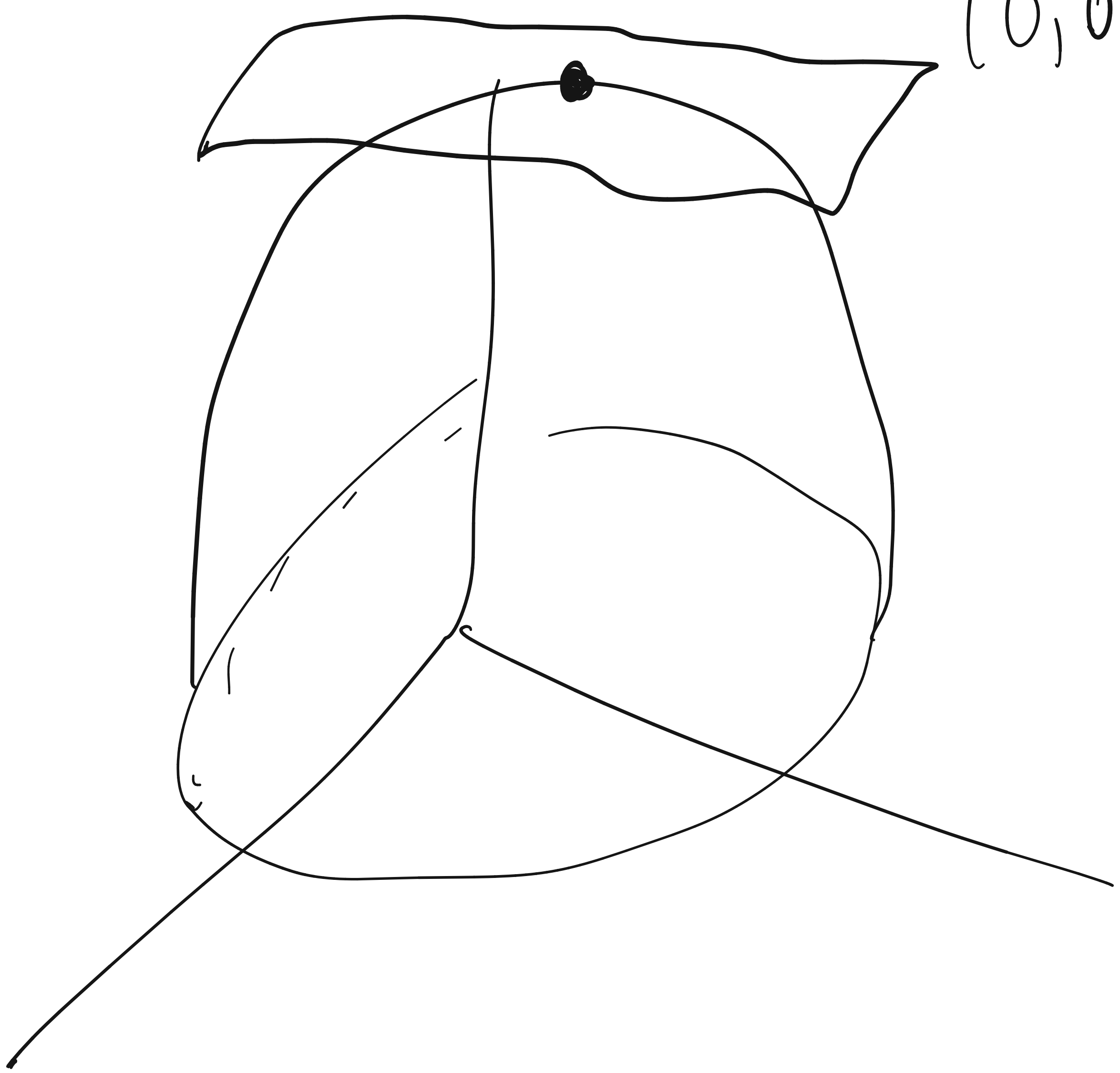
$f_y = x^2$ $f_y(1,2) = 1^2 = 1$

$$\boxed{z = 4(x-1) + 1(y-2) + 2}$$

https://www.math3d.org/7kBXnWWPp

$f = \sqrt{1-x^2-y^2}$ $\leftarrow$ graph of upper hemisphere of radius 1

tangent plane @ $(0,0)$.



$(0,0,1)$

$1 = f(0,0)$

<u>Note</u>: this plane is parallel to x-y plane!

So <u>Expect</u> Eqn of tangent plane to be $z = 1$.

let's verify this:

$$f(x,y) = \sqrt{1-x^2-y^2}$$

$$z = 0(x-0) + 0(y-0) + 1$$
$$= 1$$

$$f_x = \frac{-2x}{2\sqrt{1-x^2-y^2}} = \frac{-x}{\sqrt{1-x^2-y^2}}$$

$$f_x(0,0) = 0.$$

$$1 = \sqrt{1-0^2-0^2}$$

$$f_y = \frac{-y}{\sqrt{1-x^2-y^2}}$$

$$f_y(0,0) = 0.$$

Vocab: the <u>linearization</u> of $f(x,y)$ is

$L(x,y) =$ "Eqn of tangent plane".

ie It's just the tangent plane, but we think of it as a function rather than a geometric object.

<u>Differentials</u>:

find a small change in $f(x)$ given a small change in x, Δx.

Δf

$\Delta f \approx f'(x)\,\Delta x$ or, in "d-notation"

$df = f'(x)\,dx$

Ex we know $f(3)=4$, $f'(3)=2$,

Estimate $f(3.1)$ using differentials.

$\Delta x = +0.1$

$\Delta f \approx f'(3)\cdot\Delta x$

$\Delta f = 2\cdot(0.1) \Rightarrow \Delta f = 0.2.$

new $f = f(3.1) = f(3) + \Delta f$

$= 4 + 0.2 = \boxed{4.2}$

$f(3.1) \approx 4.2$.

Differentials in 3D:

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy$$

$$\Delta f \approx f_x(x_0, y_0)\Delta x + f_y(x_0, y_0)\Delta y$$

$\underline{Ex}$ $f(x,y) = x^2 y$

Use differentials to Estimate $f(2.9, 3.1)$

① Compute the differential df @ $(3,3)$

$$df = f_x\,dx + f_y\,dy$$

$$= 2xy\,dx + x^2 dy$$

$$df = 2\cdot 3\cdot 3\,dx + 3^2 dy$$

$$= 18dx + 9dy$$

② use df to Estim $f(2.9, 3.1)$.

$dx = -0.1$,

$dy = +0.1$

$$df = -1.8 + 0.9$$

$$= -0.9$$

new f = old f + df

$$f(3,3) = 3^2 \cdot 3 = 27$$

new $f = 27 - 0.9 = \boxed{26.1}$

$$f(2.9, 3.1) \approx 26.1$$