Today: Chain Rule:

In Calc 1: Chain rule says

$$\frac{d}{dx}\left(f(u(x))\right) = \frac{df}{du}\frac{du}{dx} = f'(u(x)) \cdot u'(x)$$

Ex $f(x) = \cos(11x^5 + 3x^4)$

inside is $u(x) = 11x^5 + 3x^4$

$f(u) = \cos(u)$

$\frac{du}{dx} = 55x^4 + 12x^3$ $\qquad$ $\frac{df}{du} = -\sin(u)$

$$\frac{df}{dx} = \frac{df}{du}\frac{du}{dx} = -\sin(u)\cdot(55x^4+12x^3)$$

$$= -\sin(11x^5+3x^4)(55x^4+12x^3)$$

This also works in 3D!

$z = f(x,y)$; $x = x(t)$, $y = y(t)$.

Think: f depends on position

x, y depend on same time parameter t

Chain Rule:

$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt}$$

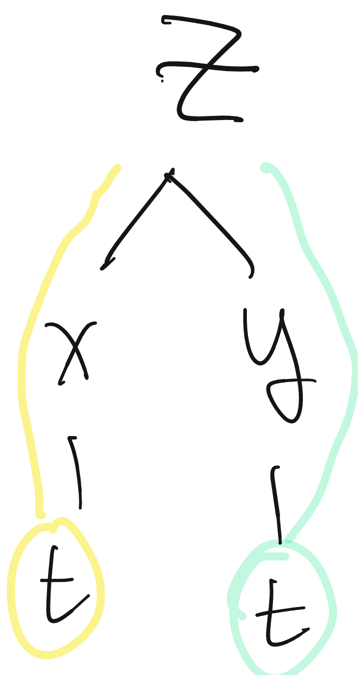
find paths from t to z,
add up paths.

Ex $z(x,y) = x^2 + x \cdot y^3.$

$x(t) = 2\cos t,\quad y(t) = 2\sin(t)$

Goal: find $\frac{dz}{dt}$ using chain rule.

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}.$$



$$\frac{\partial z}{\partial x} = 2x + y^3 \qquad \frac{dx}{dt} = -2\sin(t)$$

$$\frac{\partial z}{\partial y} = 3xy^2 \qquad \frac{dy}{dt} = 2\cos(t).$$

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

$$= (2x+y^3)(-2\sin t) + (3xy^2)(2\cos t)$$

$$= \boxed{\left[2(2\cos t) + (2\sin t)^3\right](-2\sin t) + 3(2\cos t)(2\sin t)^2 \cdot (2\cos t).}$$

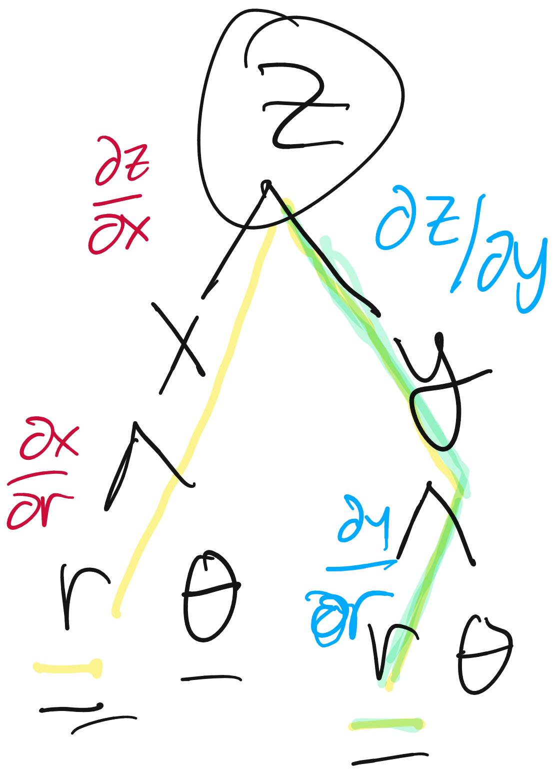
Q How is this useful? Physical applications exist!

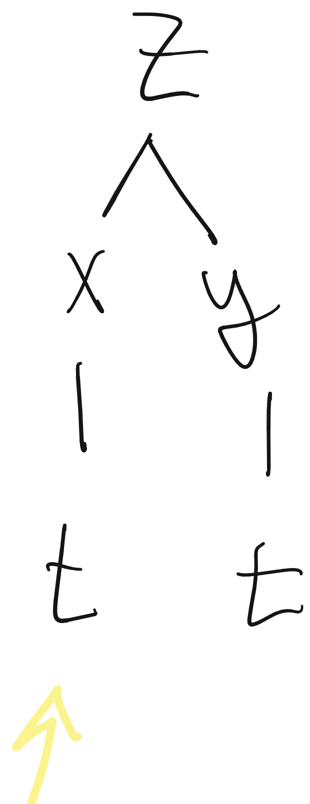
$x(0) = 2, \quad y(0) = 0.$

$f(x,y)$

$x = x(r,\theta)$

$y = y(r,\theta)$





$$\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial r}$$

only 1 variable in bottom layer : $\frac{d\,\_\_}{d\,\_\_}$

$z = x - 2xy$

find $\frac{\partial z}{\partial r}\Big|(r = 3, \theta = \pi/4)$

$x = r\cos\theta$

$y = r\sin\theta.$

$$\boxed{\begin{aligned} &\frac{\partial x}{\partial r} = \cos\theta \\ &\partial y/\partial r = \sin\theta. \\ &\frac{\partial z}{\partial y} = -2x; \\ &\frac{\partial z}{\partial x} = 1-2y; \end{aligned}}$$

$$\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial r}$$

$$\frac{\partial z}{\partial r} = (1-2y)\cos\theta + (-2x)\sin\theta$$

$$= (1-2r\sin\theta)\cos\theta + (-2r\cos\theta)\sin\theta.$$

$$= \cos\theta - 2r\sin\theta\cos\theta - 2r\cos\theta\sin\theta$$

$$= \boxed{\cos\theta - 4r\cos\theta\sin\theta} = \frac{\partial z}{\partial r}$$

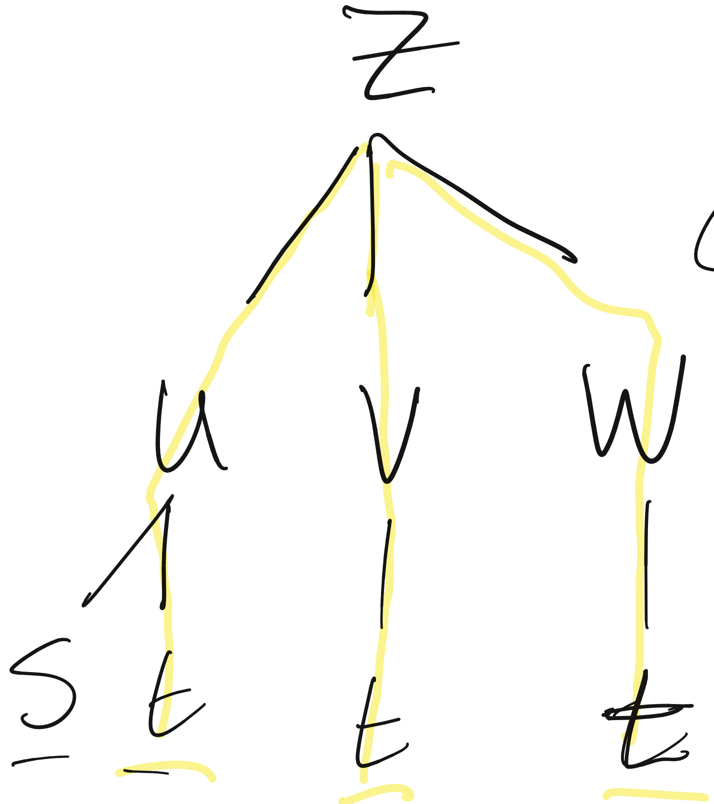
$r = 3, \ \theta = \pi/4.$

$\cos \pi/4 = \sqrt{2}/2,$

$\sin \pi/4 = \sqrt{2}/2$

$$\frac{\sqrt{2}}{2} - 4 \cdot 3 \cdot \left(\frac{\sqrt{2}}{2}\right)^2 = \frac{\sqrt{2}}{2} - 12 \cdot \frac{2}{4}$$

$$= \frac{\sqrt{2}}{2} - 6.$$



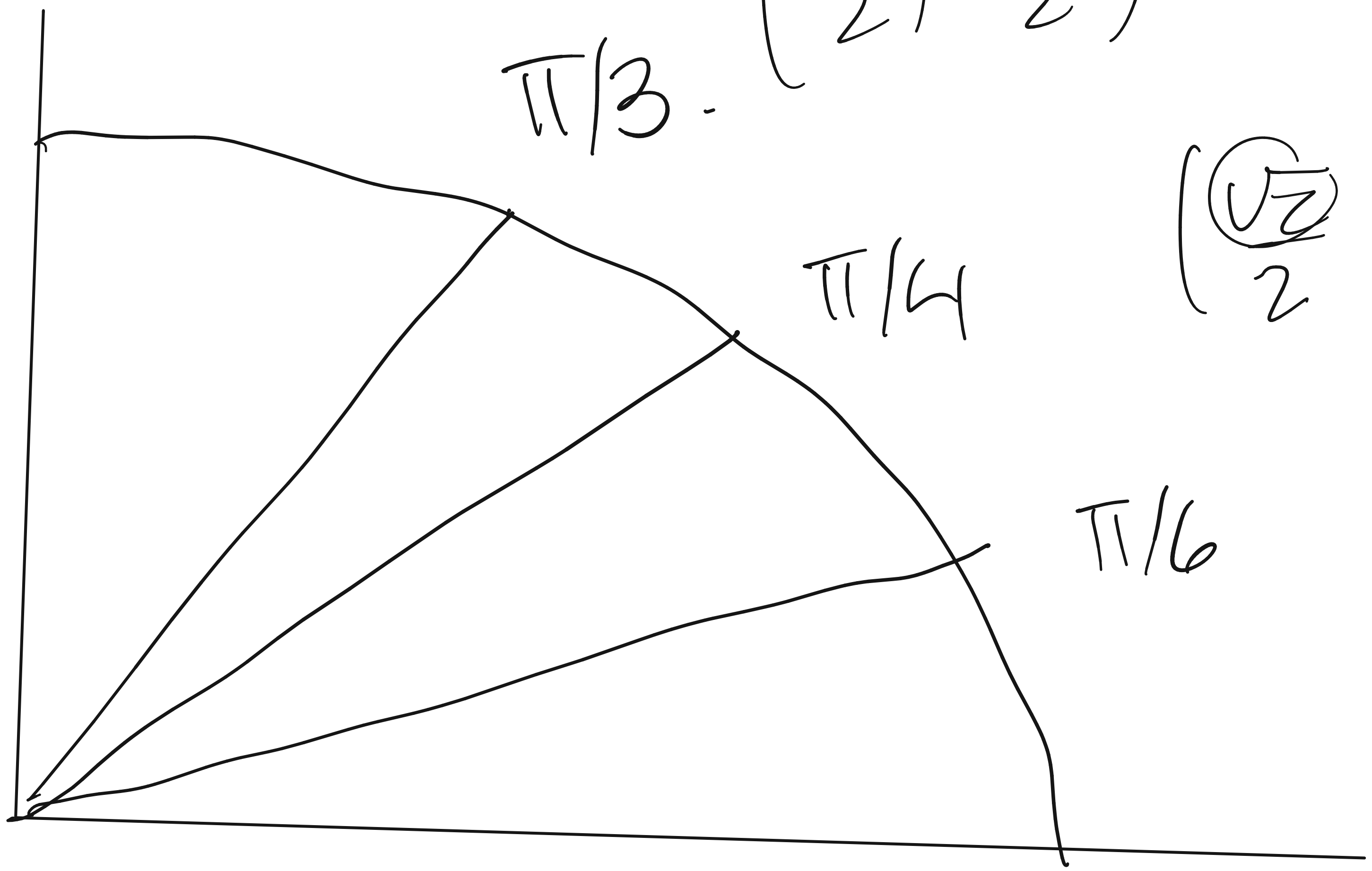
$$z = z(u, v, w)$$

$$u = u(s, t)$$

$$v = v(t)$$

$$w = w(t).$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial t} + \frac{\partial z}{\partial v} \frac{dv}{dt} + \frac{\partial z}{\partial w} \frac{dw}{dt}$$



$\pi/3 \quad \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$

$\pi/4 \quad \left(\frac{\sqrt{2}}{2}, \sqrt{2}/2\right)$

$\pi/6 \quad \left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$