10.6 (part 1) The directional Derivative

<u>Motivation:</u> why can't we take the derivative in other directions ie other than the $+x$, or $+y$ directions.

f_x = deriv. in $+x$-dir.

f_y = deriv. in $+y$-direction.

How do we find the deriv in another direction?

Setup: $\hat{u} = \langle u_1, u_2 \rangle$ $\quad (u_1^2 + u_2^2 = 1.)$

$f(x,y)$ a function $\quad (a,b)$ a point.

Def'n: the directional derivative of $f(x,y)$

in the direction $\hat{u}$ at the point (a,b) is

$\boxed{\text{has to be a unit vector}}$

$$D_{\hat{u}} f(a,b) = \lim_{h \to 0} \frac{f(a+hu_1, b+hu_2) - f(a,b)}{h}$$

unit vectors only!

Aside:

$$D_{\vec{u}} f(a,b) = \lim_{h \to 0} \frac{1}{h\|\vec{u}\|}\left[f(a+hu_1, b+hu_2) - f(a,b)\right].$$

"Frechet Derivative":

Theorem: $D_{\hat{i}} f(a,b) = f_x(a,b)$

$D_{\hat{j}} f(a,b) = f_y(a,b)$.

Ex $f(x,y) = -4x^2 + 2y^2$.

$D_{\hat{u}} f(1,1)$ where $\hat{u} = \langle \frac{1}{\sqrt{2}}, -1/\sqrt{2} \rangle$.

① Check: verify that $\hat{u}$ is a unit vector.

② $f(1+h\frac{1}{\sqrt{2}}, 1-h\frac{1}{\sqrt{2}})$, $f(1,1)$

$f(1,1) = -2.$

$$f(\sim, \sim) = -4(1+h/\sqrt{2})^2 + 2(1-h/\sqrt{2})^2$$
$$= -h^2 - 6\sqrt{2}h - 2.$$

$$f(1+h/\sqrt{2}, 1-h/\sqrt{2}) - f(1,1) = -h^2 - 6\sqrt{2}h - 2 + 2$$
$$= -h^2 - 6\sqrt{2}h$$

$$\lim_{h\to 0} \frac{1}{h}\left(-h^2 - 6\sqrt{2}h\right) = \lim_{h\to 0} -h - 6\sqrt{2} = \boxed{-6\sqrt{2}}$$

$$D_{\langle 1/\sqrt{2},\, -1/\sqrt{2} \rangle} f(1,1) = -6\sqrt{2}.$$

Scalar value!

<u>Theorem:</u> Let $f(x,y)$ be a function such that $f_x(a,b)$, $f_y(a,b)$ exist @ a point (a,b).

Let $\hat{u}$ be a unit vector $\hat{u} = \langle u_1, u_2 \rangle$.

Then: $\boxed{D_{\hat{u}} f(a,b) = f_x(a,b)\, u_1 + f_y(a,b)\, u_2.}$

Geometrically, this looks like a dot product!

$$\langle f_x(a,b), f_y(a,b) \rangle \cdot \langle u_1, u_2 \rangle.$$

Ex $f(x,y) = 3xy - x^2y^3$.

Goal is find $D_{\vec{v}}(1,-1)$ where $\vec{v} = \langle 2, 3 \rangle$.

① first, find $f_x(1,-1)$, $f_y(1,-1)$

$f_x = 3y - 2xy^3$ $\quad f_x(1,-1) = -3+2 = -1$

$f_y = 3x - 3x^2y^2$. $f_y(1,-1) = 0$

② Convert $\vec{v} = \langle 2,3 \rangle$ into unit vector $\hat{v}$

$$\|\vec{v}\| = \sqrt{2^2+3^2} = \sqrt{13}$$

$$\hat{v} = \frac{\vec{v}}{\|\vec{v}\|} = \langle 2/\sqrt{13}, 3/\sqrt{13} \rangle$$

③ Combine the ingredients

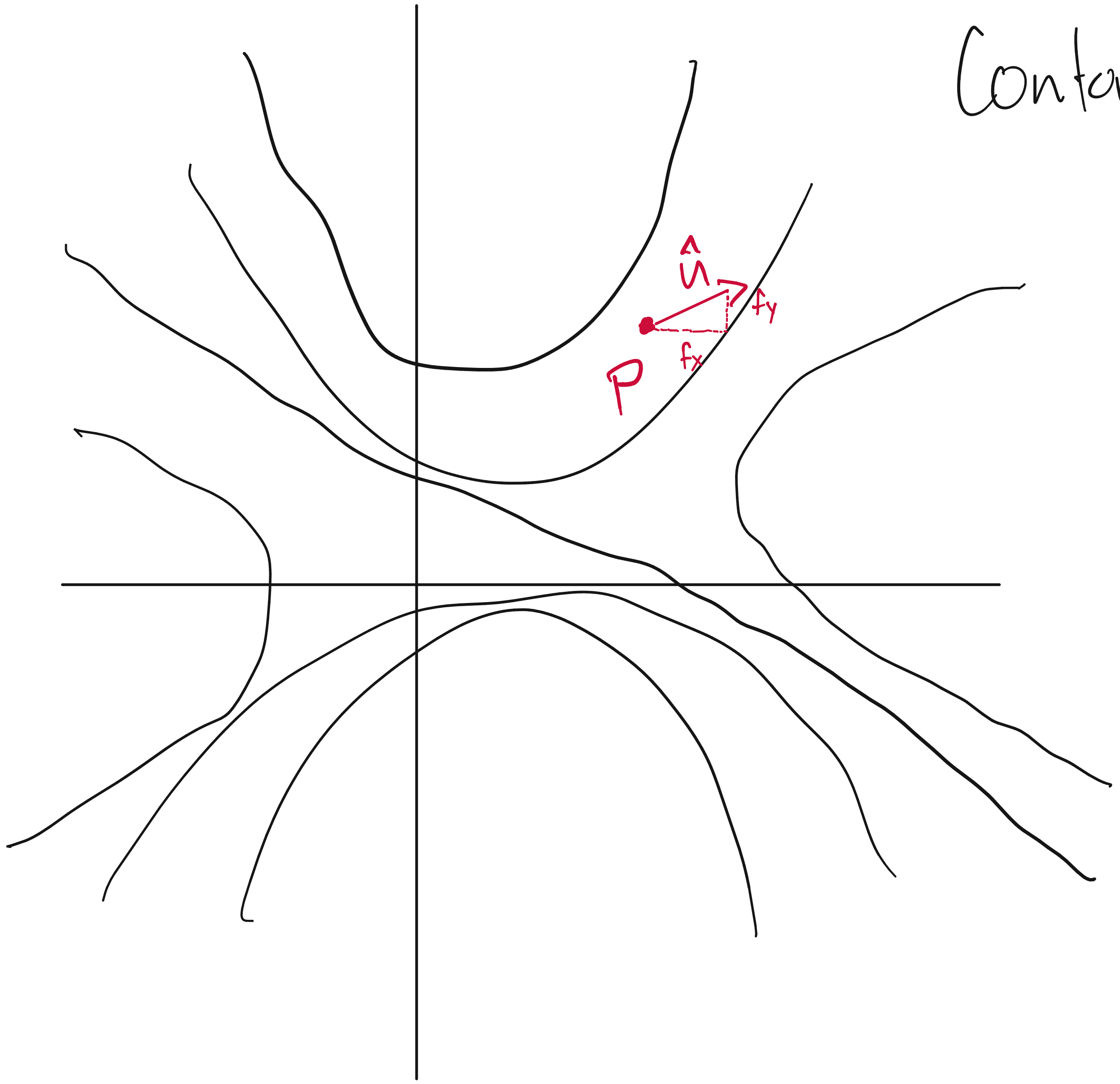
$$D_{\hat{v}} f(1,-1) = f_x u_1 + f_y u_2 = -1\left(2/\sqrt{13}\right) + 0\left(3/\sqrt{13}\right)$$

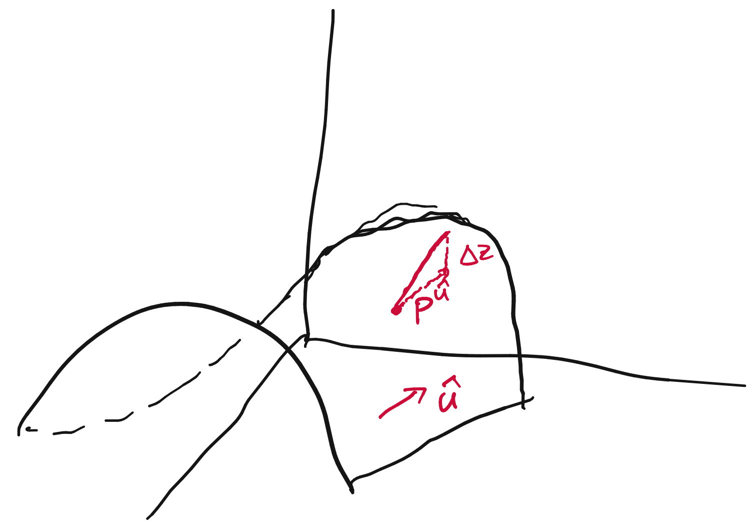
$= -2/\sqrt{13}$

Summary of Steps:

1. find $f_x(a,b)$, $f_y(a,b)$
2. Unit-vectorify your direction (if needed)
3. $D_{\hat{v}}f = f_x v_1 + f_y v_2$ & Evaluate.

Contour plot of f(x,y)





Recall: $D_{\hat{u}} f(a,b) = \underbrace{\langle f_x(a,b), f_y(a,b) \rangle}\cdot \hat{u}$.

Def'n the gradient of $f(x,y)$ is the vector-valued function

a function that takes in points (a,b) and spits out a vector $\vec{f}(a,b) = \langle f_1(a,b), f_2(a,b) \rangle$.

$$\boxed{\vec{\nabla} f(x,y) = \langle f_x(x,y), f_y(x,y) \rangle}$$

colloquially: "del".

∇ = "nabla" $\longrightarrow$ word for an ancient type of harp.

With this notation,

$$(*) \rightarrow \boxed{D_{\hat{u}} f(a,b) = \vec{\nabla} f(a,b) \cdot \hat{u}} \leftarrow (*)$$

Properties of gradient:

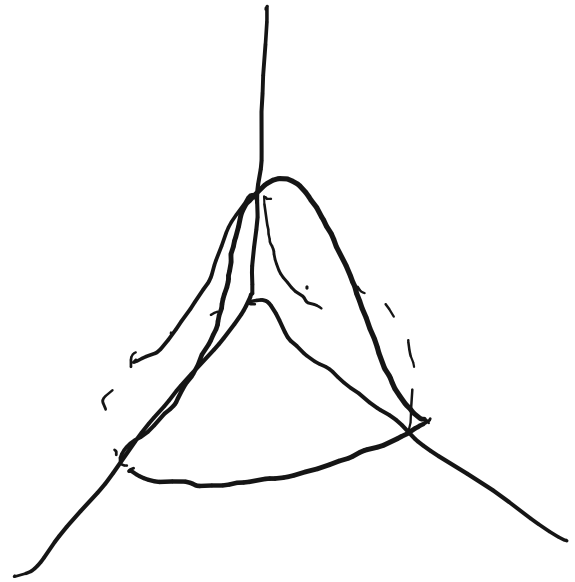
1. The gradient of $f(x,y)$ @ (a,b) points in the dir. of greatest local ascent.

   ie $\vec{\nabla} f(a,b)$ points towards higher z-values

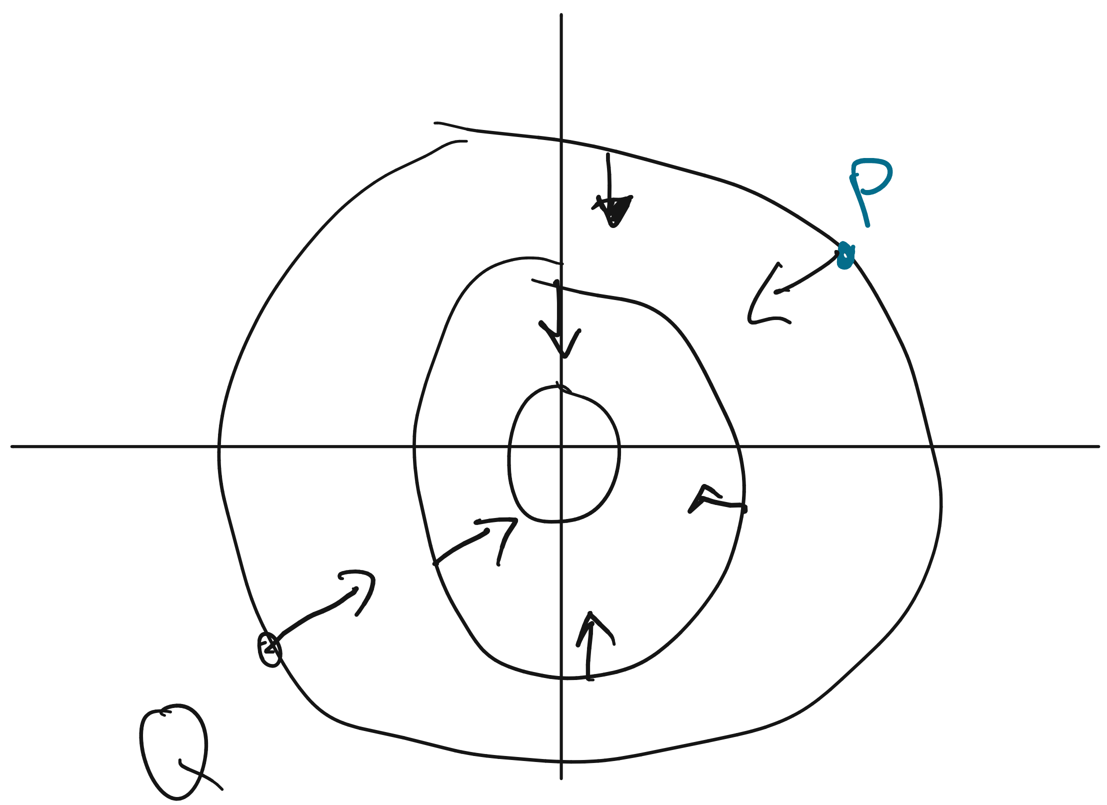
   "Steepest ascent"

2. $\nabla f(a,b)$ is perpendicular to the level set of $f(x,y)$ containing the point (a,b).

$f = e^{-x^2-y^2}$

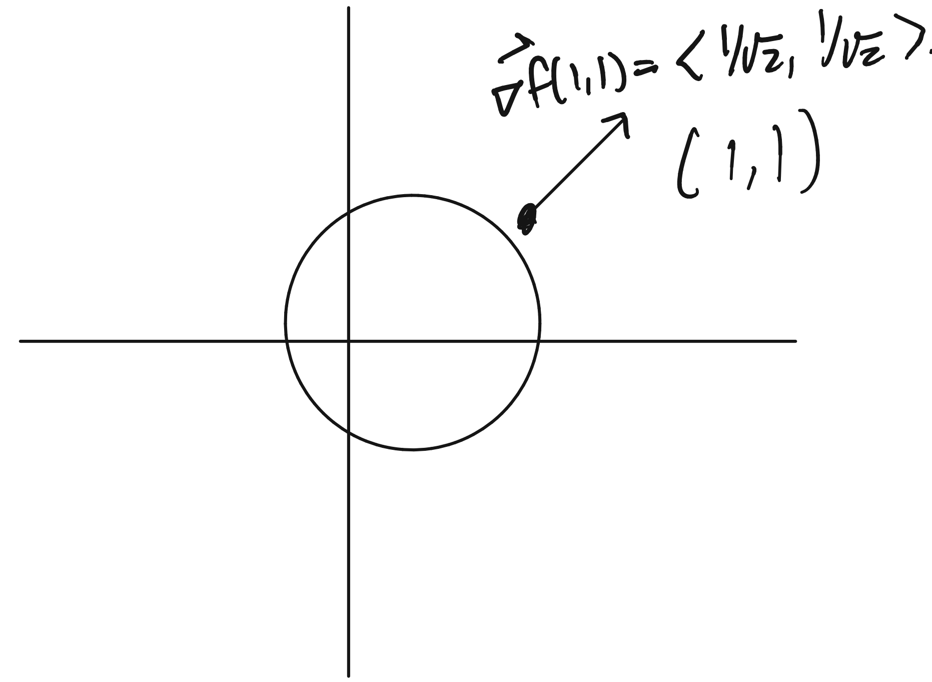


Contour plot:



$\vec{\nabla} f(P)$

$\underline{Ex}$ $f(x,y) = \sqrt{x^2+y^2}$



$$\nabla f = \langle f_x, f_y \rangle$$

$$f_x = 2x \cdot \frac{1}{2}(x^2+y^2)^{-1/2} = \frac{x}{\sqrt{x^2+y^2}}$$

$$f_y = y/\sqrt{x^2+y^2}$$

$$\vec{\nabla} f(x,y) = \left\langle \frac{x}{\sqrt{x^2+y^2}}, y/\sqrt{x^2+y^2} \right\rangle.$$

$$\nabla f(1,1) = \langle 1/\sqrt{2}, 1/\sqrt{2} \rangle$$

Note: the differential and the gradient contain the same information!

$$df = f_x\,dx + f_y\,dy$$

$$\vec{\nabla} f = \langle f_x, f_y \rangle.$$

Note: is that ∇f makes sense when

f is a function of >2 variables too!

Ex $f(x,y,z) = xyz - yz^2$.

$$\vec{\nabla} f = \langle f_x, f_y, f_z \rangle.$$

$$\vec{\nabla} f = \langle yz, xz - z^2, xy - 2yz \rangle.$$