Yesterday: Directional derivatives & the Gradient.

$$D_{\hat{u}} f(a,b) = f_x(a,b)\,u_1 + f_y(a,b)\,u_2 \qquad \hat{u} \qquad \nabla f$$

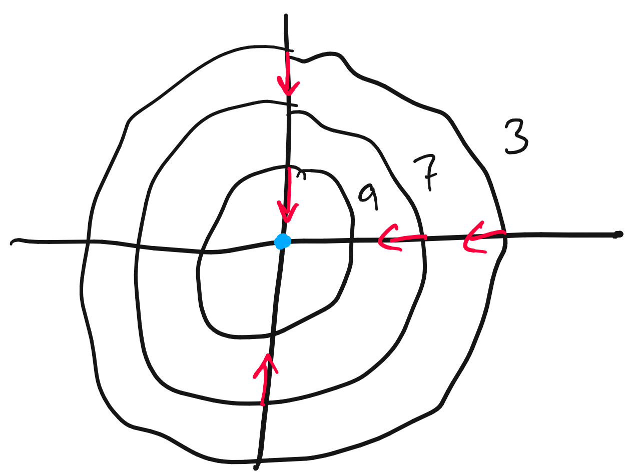
where $\hat{u} = \langle u_1, u_2 \rangle$ unit vector.

Scalar.

$$\vec{\nabla} f(x,y) = \langle f_x(x,y),\ f_y(x,y) \rangle$$

a vector.

Gradient tells us the direction of greatest ascent.



• = Critical point.

• = gradient vectors.

gradient vectors are perpendicular to contours.

Notice: $D_{\hat{u}} f(x,y) = \vec{\nabla} f \cdot \hat{u}$

So... recall the dot product (Cosine version)

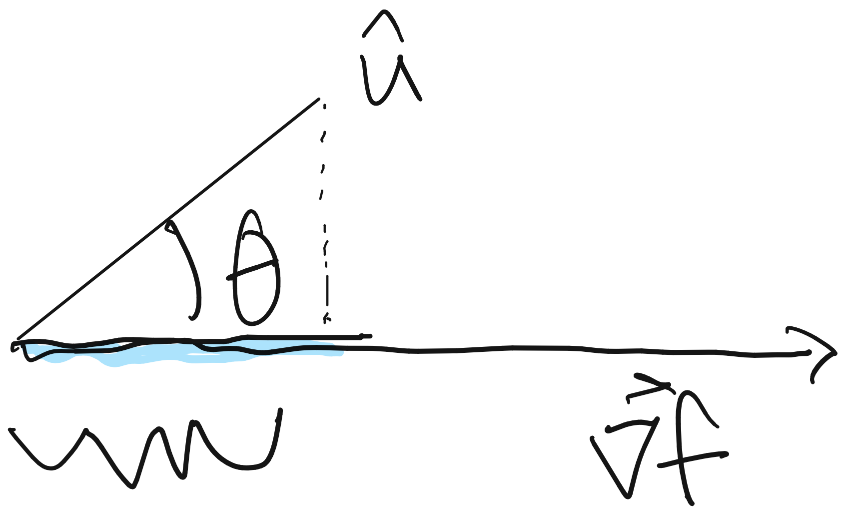
$$\vec{v} \cdot \vec{w} = \|\vec{v}\| \|\vec{w}\| \cos\theta$$

Applying this here, we see:

$\hat{u}$ unit vector!

$$D_{\hat{u}} f = \vec{\nabla} f \cdot \hat{u} = \|\vec{\nabla} f\| \cdot \|\hat{u}\| \cdot \cos\theta$$

$$= \|\vec{\nabla} f\| \cos\theta.$$

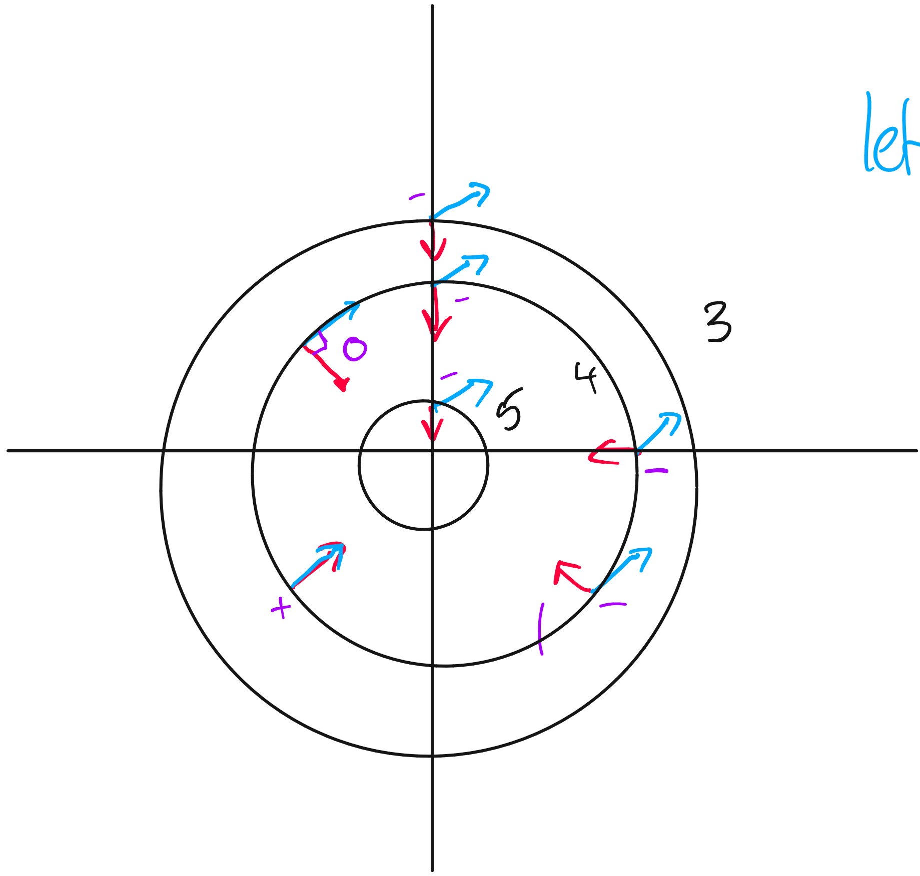


this length represents $D_{\hat{u}} f$.

$$D_{\hat{u}} f = \vec{\nabla} f \cdot \hat{u}$$
$$= \|\nabla f\| \cos\theta.$$

If $\hat{u} \perp \nabla f$ then $D_{\hat{u}} f = 0$.

This gives us a way to estimate the sign of $D_{\hat{u}} f$ using our geometric intuition



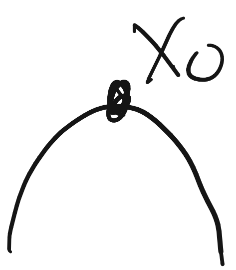
let $\hat{u} = \langle 1/\sqrt{2}, 1/\sqrt{2} \rangle$.

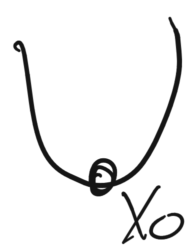
Purple denote the sign of $D_{\hat{u}} f(\text{pt.})$.

§10.7: "Optimization" $\longrightarrow$ "Critical Point Theory".

In Calc 1, a critical point of a function is an x-value x_0 st $f'(x_0) = 0$.

Crit pts come in 2 varieties in 1-D:





local maxes — local mins

$-x^2$ — $+x^2$.

This same idea works in functions of two variables!

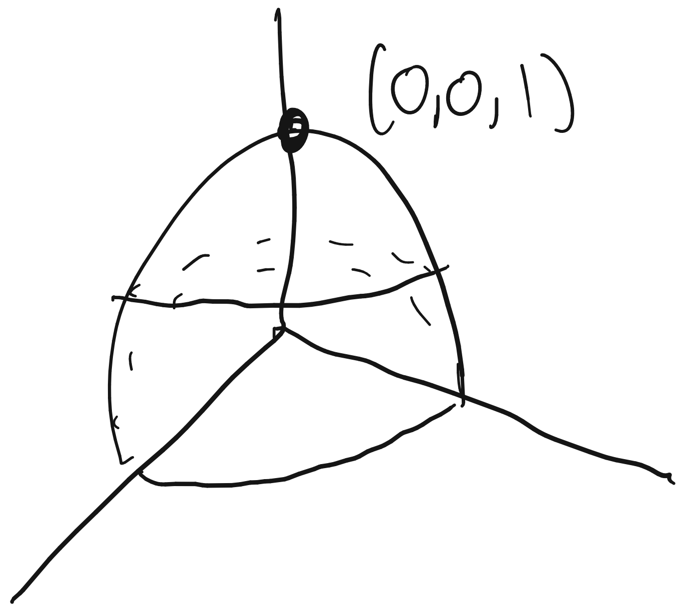
<u>Def</u>: Let $f(x,y)$ be a func. of 2 vars.

We say that (a,b) is a <u>critical point</u>

of $f(x,y)$ if $\vec{\nabla} f(a,b) = \vec{0}$

ie $f_x(a,b) = 0$ <u>AND</u> $f_y(a,b) = 0$.

Ex Claim is $f(x,y) = \sqrt{1-x^2-y^2}$ has a critical point @ $(0,0)$



$\boxed{\text{local max.}}$

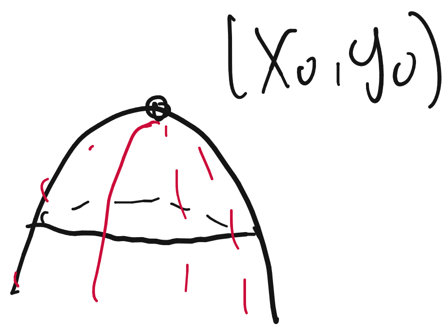
$$\nabla f = \left\langle \frac{-x}{\sqrt{1-x^2-y^2}}, \frac{-y}{\sqrt{1-x^2-y^2}} \right\rangle$$

$$\nabla f(0,0) = \langle 0, 0 \rangle = \vec{0}$$

So $(0,0)$ is a Crit. Pt. of $f(x,y)$.

A Gallery of Critical Points:

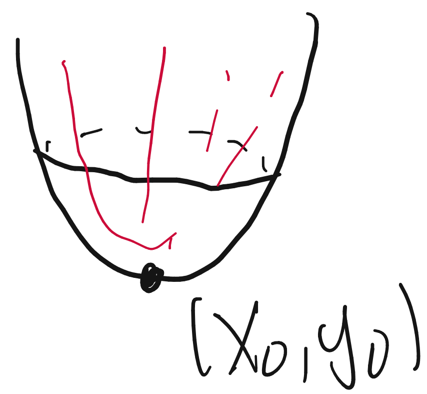
local max



Ex: $-x^2 - y^2$

@ $(0,0)$

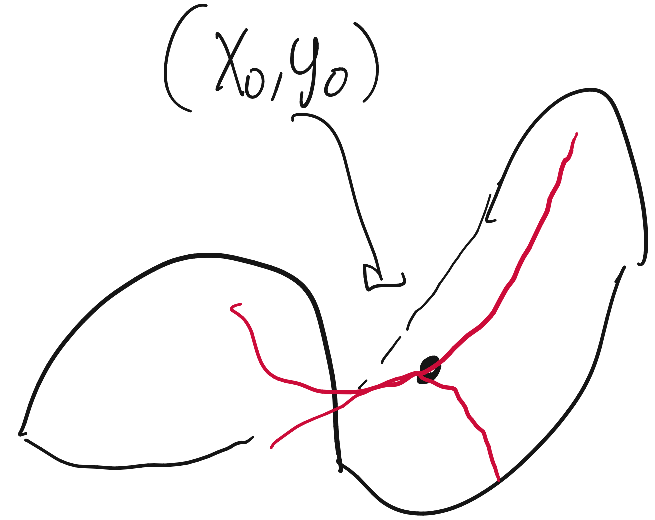
local min:



Ex: $x^2 + y^2$

@ $(0,0)$

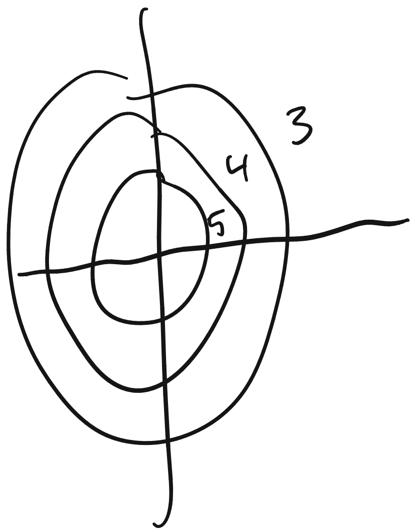
Saddle Point.



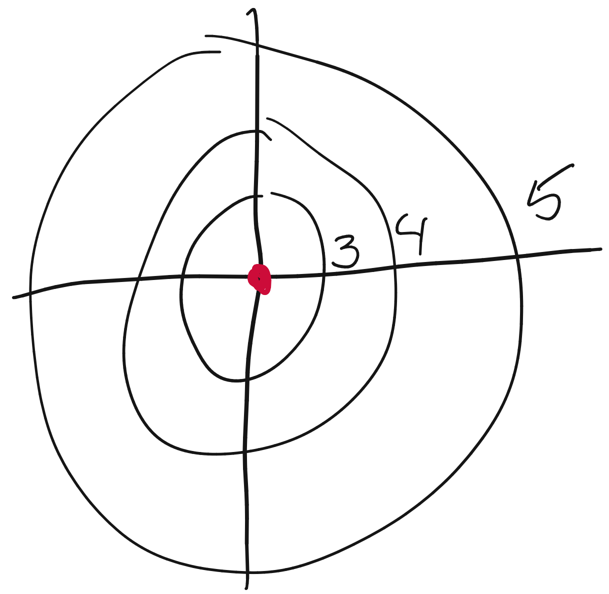
Ex: $x^2 - y^2$ @ $(0,0)$.

On a contour diagram:

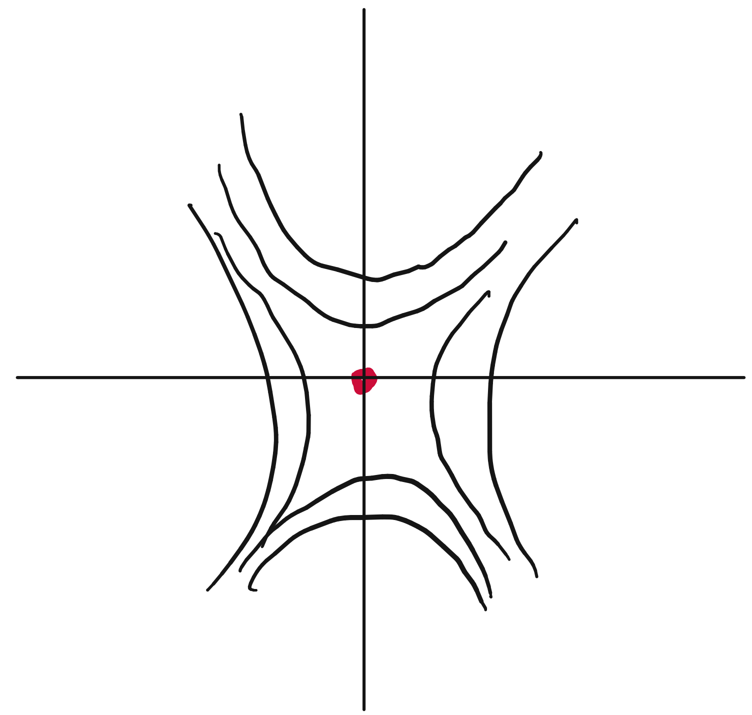
local max



local min



Saddle Point:



Finding Crit. pts:

Ex $f(x,y) = 2xy + 4x - 2y + 3$.

Method ① find $\vec{\nabla} f$

② Set $\vec{\nabla} f = 0$

③ Solve for your crit pts.

$$\nabla f = \langle 2y+4,\ 2x-2 \rangle = \langle 0, 0 \rangle$$

$$\begin{cases} 2y+4=0 & \Rightarrow y=-2 \\ 2x-2=0. & \Rightarrow x=1. \end{cases}$$

Only crit. pt is @ $\boxed{(1,-2).}$

Check: $\langle 2\cdot(-2)+4,\ 2(1)-2 \rangle = \langle 0, 0 \rangle.$

Ex $f(x,y) = 3x^3 + y^2 - 9x + 4y$

$$\vec{\nabla} f = \langle 9x^2-9,\ 2y+4 \rangle = \langle 0,0 \rangle$$

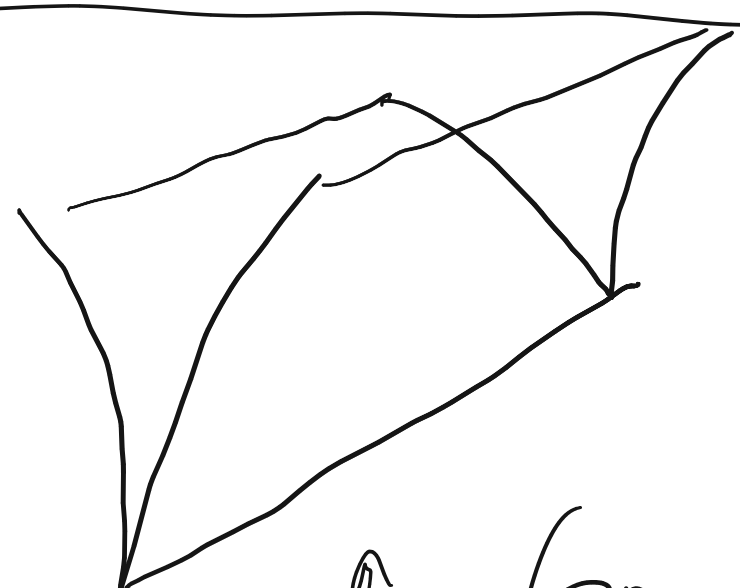
$$\begin{cases} 9x^2-9=0 \\ 2y+4=0. \end{cases} \Rightarrow \begin{array}{l} x=\pm 1 \\ y=-2. \end{array}$$

two potential crit points: $(1,-2)$ ✓

$(-1,-2)$ ✓

$$\vec{\nabla} f(1,-2) = \langle 9\cdot(1)^2 - 9,\ 2(-2)+4 \rangle = \langle 0,0 \rangle$$

$$\nabla f(-1,-2) = \langle 9(-1)^2 - 9,\ 2(-2)+4 \rangle = \langle 0,0 \rangle$$



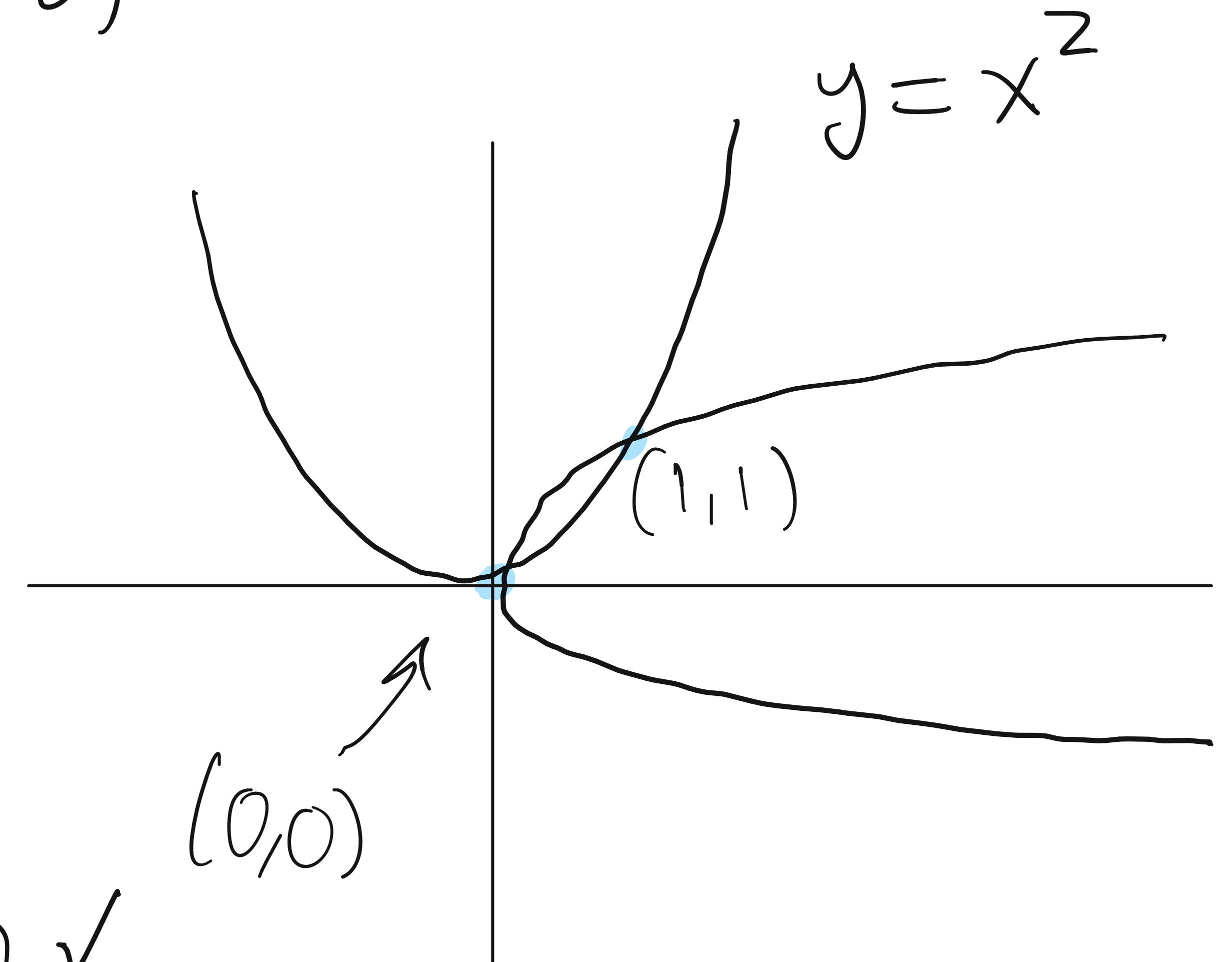
$(\sqrt[3]{x})\, y$

∞'ly many critical points

Ex $f(x,y) = x^3 + y^3 - 3xy$.

$$\vec{\nabla} f = \langle 3x^2 - 3y,\ 3y^2 - 3x \rangle = \langle 0,0 \rangle.$$

$$\Rightarrow \begin{cases} x^2 = y \\ y^2 = x. \end{cases}$$



Claim: $\boxed{(0,0),\ (1,1)}$ are both crit. pts of $f(x,y)$.