Last week: Optimization

→ main objective: find & classify all critical points of $f(x,y)$.

Two big Eqn's / formulas:

① Def'n of crit point $\vec{\nabla} f = \vec{0}$

② Discriminant: $D = f_{xx} \cdot f_{yy} - (f_{xy})^2$

→ helps classify crit points.

ie tells what type of crit pt. we have.

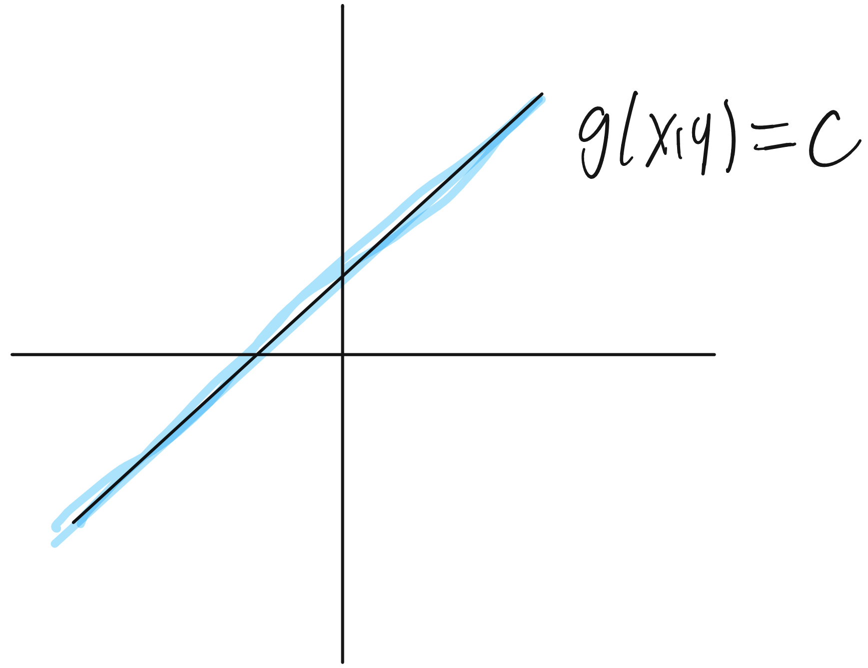
Another setup: Given a function $f(x,y)$ find mins/maxes of f subject to some constraint

$g(x,y) = C$.

level curve

Geometrically: look for mins/maxes of $f(x,y)$ on the curve $g(x,y) = C$.

Sketch



The Method of Lagrange Multipliers (1870's?)

Setup: Function $f(x,y)$ that we want to optimize

Constraint $g(x,y) = C$.

What you do: Introduce a new Variable

$\lambda \leftarrow$ lambda & form the following

System of Eqn's:

$$\begin{cases} ① \quad g(x,y) = C \\ ② \quad \vec{\nabla} f = \lambda \vec{\nabla} g. \end{cases}$$

Note: Sol'ns (x,y,λ) to the Method of Lagr. mults are NOT generally critical points of $f(x,y) \Rightarrow$ Can't appeal to 2^{nd} deriv test.

Solve for x,y,λ and plug back into f.

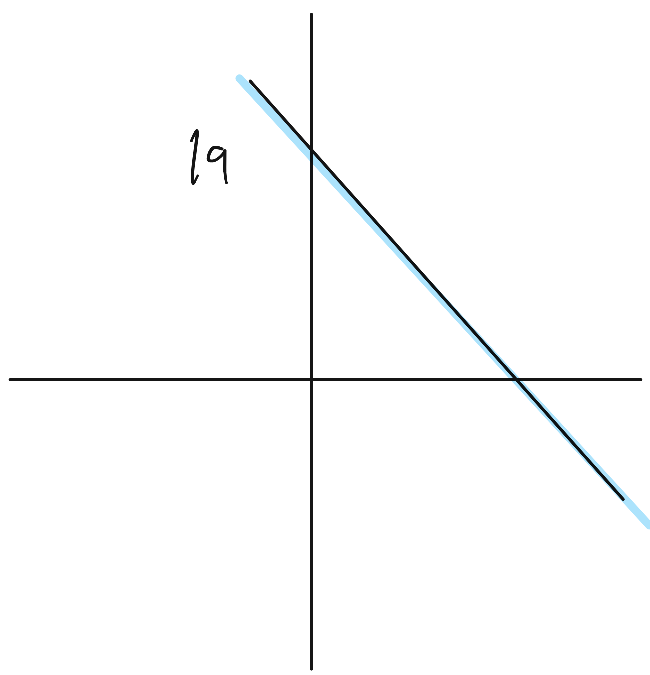
Ex

Find the minimum value of

$f(x,y) = x^2 + y^2$ subject to the constraint

$4x + 6y = 19.$ (Using Lagrange Multipliers)

$g(x,y) = C.$



$$\begin{cases} 4x + 6y = 19 \\ \vec{\nabla} f = \lambda \vec{\nabla} g \end{cases}$$

$$2x = 4\lambda$$
$$2y = 6\lambda$$

*

$$\vec{\nabla} f = \langle 2x, 2y \rangle$$
$$\vec{\nabla} g = \langle 4, 6 \rangle$$

Substitute!

$$x = 2\lambda$$
$$y = 3\lambda$$

$$4(2\lambda)+6(3\lambda)=19$$

$$8\lambda+18\lambda=19$$

$$26\lambda=19 \qquad \boxed{\lambda=19/26}$$

$$\Rightarrow \boxed{\begin{aligned} x &= 19/13 \\ y &= 57/26 \end{aligned}}$$

this is our minimum value for $f(x,y)$ subject to $g(x,y)=C$.

Check $f\left(\frac{19}{13}, 57/26\right) = \frac{361}{52} \approx \boxed{6.942.}$

How do we see that this is a minimum?

$$x' = \frac{19}{13} + \frac{\varepsilon}{4} \qquad (\text{think: } \varepsilon = 0.001)$$

$$y' = \frac{57}{26} - \frac{\varepsilon}{6}. \qquad \text{then } g(x', y') = C. \quad \underline{\text{Still}}$$

<u>Claim</u>: $f(x', y') > f(x, y)$

How do we interpret λ?

"Physically" λ tells us how much our max/min value of $f(x,y)$ changes when we adjust the constraint $g(x,y) = C$.

(In Econ, λ is called "the shadow price")

(WW 10.8 #5)

The max value of $f(x,y)$ subject to constant $g(x,y) = 220$ is found to be 6700. The method of L. multipliers gives $\lambda = 20$.

Find an approximate value of f_{max} subject to $g(x,y) = 217$.

$$\Delta f_{max} = \lambda \cdot \Delta C = 20 \cdot -3 = -60.$$

Change in constraint. $= -3$ (here)

$$f_{max}^{new} \approx f_{max}^{old} + \Delta f_{max} = 6700 - 60 = 6640$$

So $f_{max}^{new} \approx 6640$

Ex $f(x,y,z) = 4x+4y+3z$

$g(x,y,z) = x^2+y^2+z^2 = 4.$

find mins/maxes of $f(x,y,z)$ subject to g.

$\vec{\nabla} f = \lambda \vec{\nabla} g$

$\vec{\nabla} f = \langle 4, 4, 3 \rangle$

$\vec{\nabla} g = \langle 2x, 2y, 2z \rangle$

$$\begin{cases} x^2+y^2+z^2=4 \\ 4 = 2x\cdot\lambda & \text{(A)} \\ 4 = 2y\cdot\lambda & \text{(B)} \\ 3 = 2z\cdot\lambda & \text{(C)} \end{cases}$$

$\frac{(A)}{(B)}: \; 1 = \frac{x}{y} \Rightarrow \boxed{y = x}$

$$\frac{C}{A} = \frac{3}{4} = \frac{z}{x} \Rightarrow \quad 3x = 4z$$

$$\Rightarrow z = \tfrac{3}{4}x$$

$$x^2 + (x)^2 + \left(\tfrac{3}{4}x\right)^2 = 4$$

$$\left(2 + \frac{9}{16}\right)x^2 = 4$$

$$x = \pm \tfrac{4}{\sqrt{14}} = y$$

$$z = \tfrac{3}{4}x = \pm \tfrac{3}{\sqrt{14}}$$

all w/ same sign.
All pos've or all neg've.

$f(4/\sqrt{14}, 4/\sqrt{14}, 3/\sqrt{14}) \approx 12.8062$

↑ max value

$f(-, -, -) \approx -12.8062$ ← min value.