§11.1 Double Integrals over Rectangles:

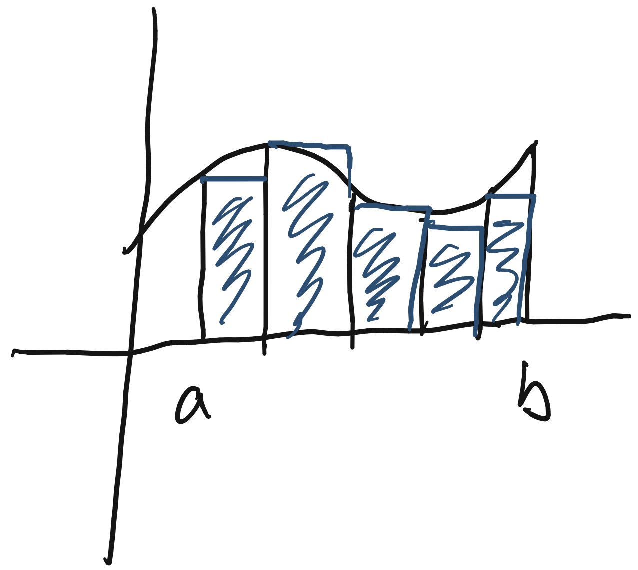
Setup: Calc 1/2 Example:

x	$f(x)$
0	3
1	7
2	9
3	11
4	13

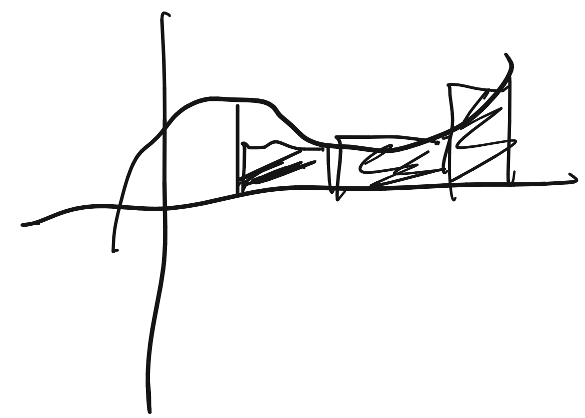
$\Delta x = 1$

Estimate $\int_0^4 f(x)\,dx$ using this table

Left:



Right:



$L = (3+7+9+11)\,\Delta x = 30$

$R = (7+9+11+13)\,\Delta x = 40$

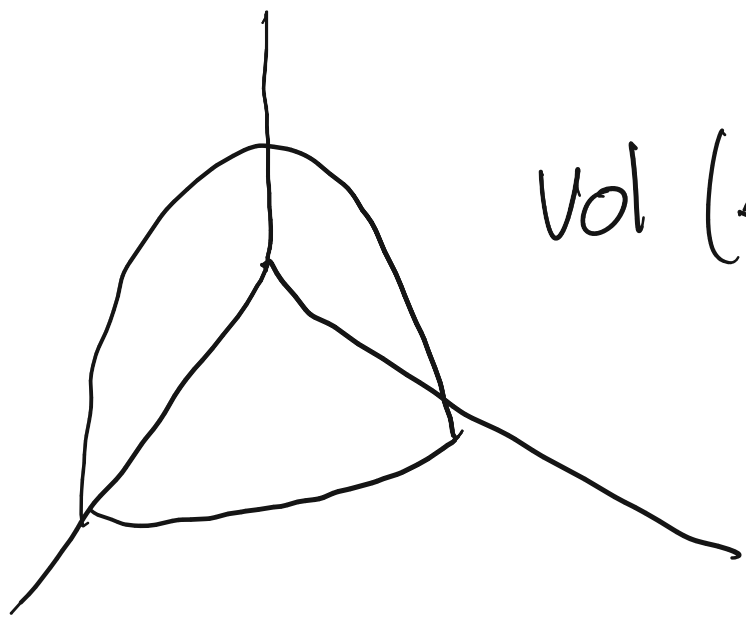
to get a better Estimate: average these two estimates!

$$M = \frac{30+40}{2} = \boxed{35} \leftarrow \cancel{\text{best}}\ \text{good}\ \text{Estm. for}$$

$$\int_0^4 f(x)\,dx.$$

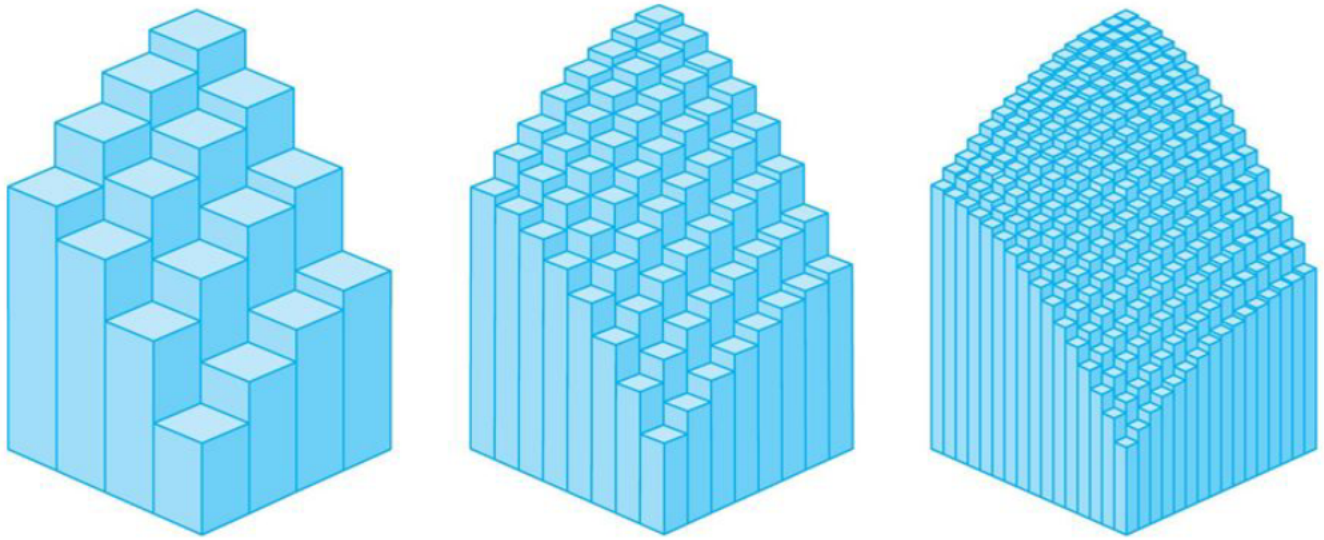
<u>Big Idea</u>: approximate Area under curve w/ rectangular strips

Motivation: Volume under a surface should be computable by an integral! (or two)



$$\text{vol}\left(\quad \right) = \iint_R f(x,y)\,dA$$

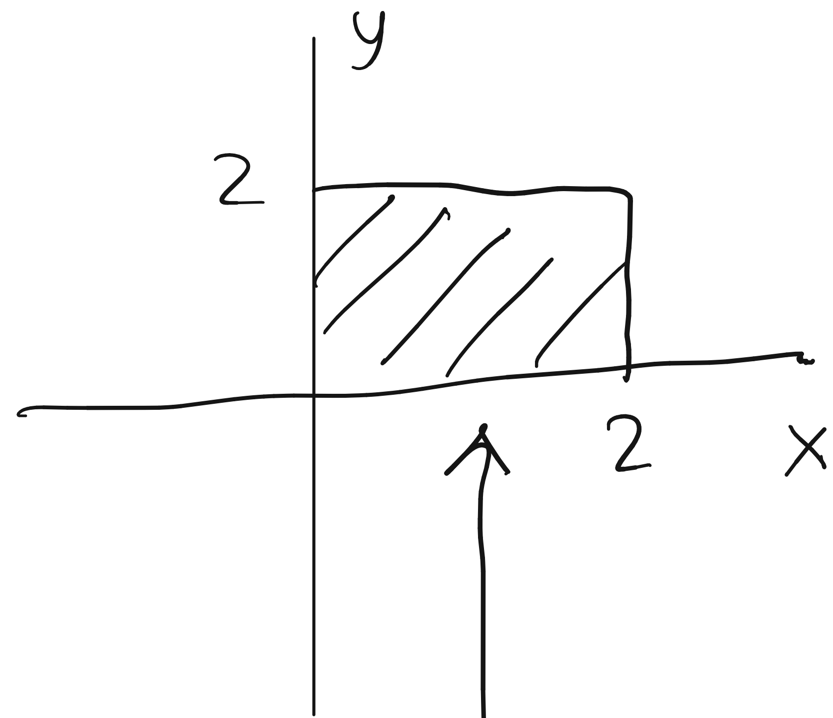
dA is a small bit of area.



(a) $m = n = 4$, $V \approx 41.5$ (b) $m = n = 8$, $V \approx 44.875$ (c) $m = n = 16$, $V \approx 46.46875$

FIGURE 8

The Riemann sum approximations to the volume under $z = 16 - x^2 - 2y^2$ become more accurate as m and n increase.



Ex find volume under graph of $f(x,y) = 3 + x + y$

when $[0 \le x \le 2,\ 0 \le y \le 2.]$

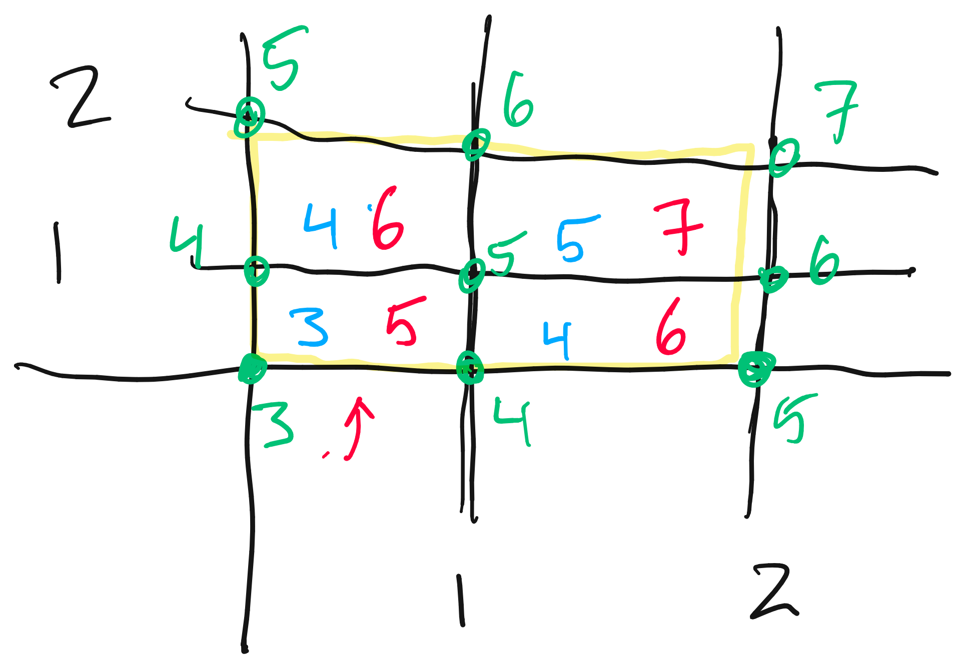
$\uparrow$

region of integration.

y/x	0	1	2
2	5	6	7
1	4	5	6
0	3	4	5

$$\Delta A = \Delta x \Delta y = 1$$

"Lattice points"



min method

max method

$$m = (3+4+4+5)\Delta A = 16$$ "Lower estimate / bound".

$M = (5+6+6+7)\Delta A = 24$ "upper estimate / bound"

Estim: $E = \dfrac{M+m}{2} = \dfrac{16+24}{2} = \boxed{20}$ ← good estimate for

$$\int_0^2 \int_0^2 f(x,y)\,dx\,dy \approx 20.$$

y-bounds. x-bounds

Def'n: let R be the rectangle

$a \leq x \leq b, \quad c \leq y \leq d.$

and let $f(x,y)$ be a function of two variables.

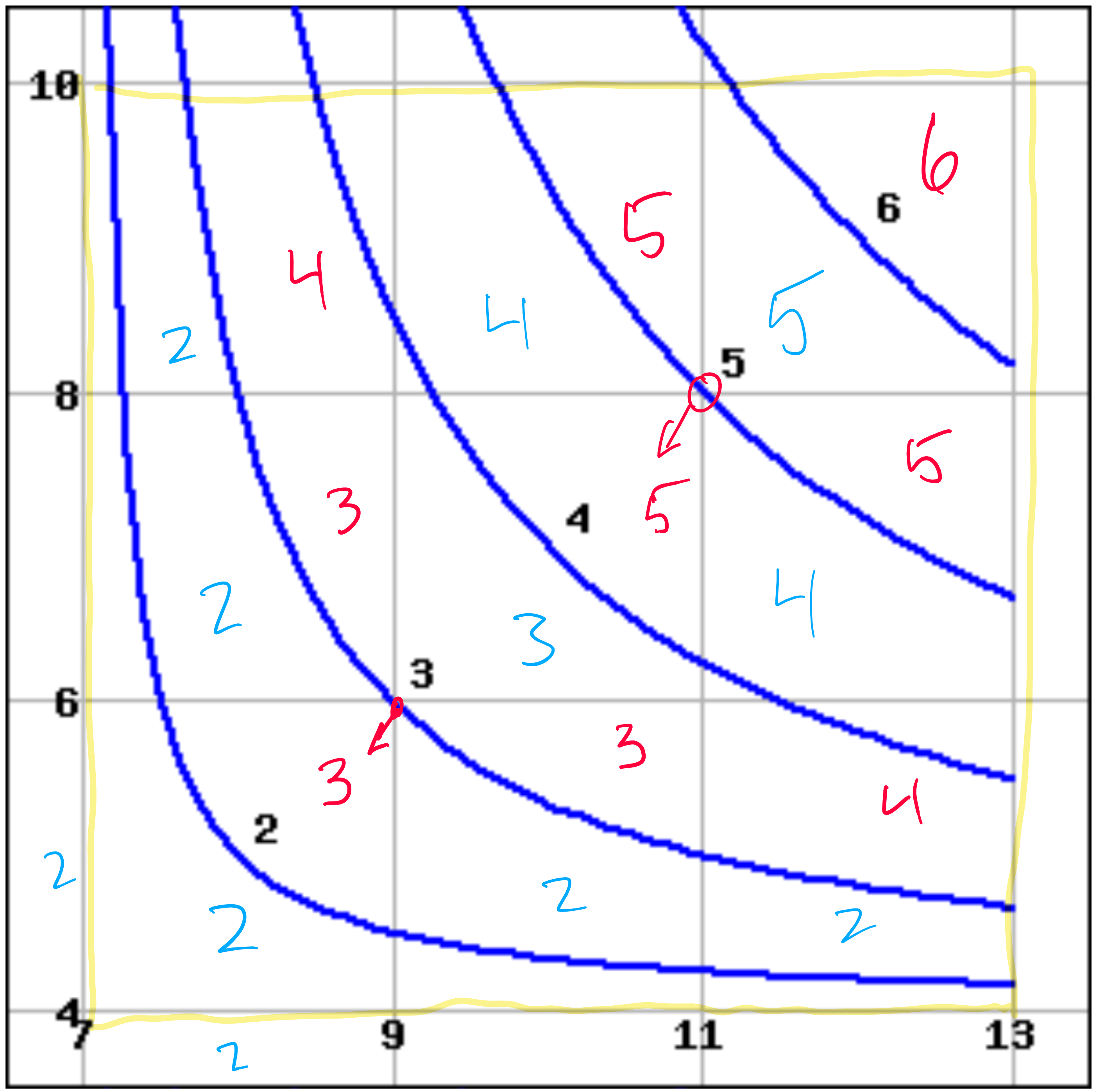
The double integral of $f(x,y)$ over the region R

is:

pick a point in each rectangle.

$$\iint_R f(x,y)\, dA = \lim_{m \to \infty} \left(\lim_{n \to \infty} \left(\sum_{i=0}^{m} \sum_{j=0}^{n} f(x_{ij}^*, y_{ij}^*)\, \Delta A \right)\right)$$

This is gross. Just do $\dfrac{\min + \text{Max}}{2}$ method!



goal compute $\iint_R g(x,y)\,dA$ over

$$m = (2+2+2+2+2+3+4+4+5)\Delta A$$

$$\Delta A = \underset{\Delta x}{2} \times \underset{\Delta y}{2} = 4$$

$$M = (4+3+3+3+4+5+5+5+6)\Delta A$$

$$\Delta A = 2\times 2 = 4.$$

Estim: $\dfrac{M+m}{2}$.

Fact: If $f(x,y)$ is continuous on the rectangle $a \le x \le b$, $c \le y \le d$ then:

① the integral $\iint_R f(x,y)\,dA$ exists.

② the average value of f over any region R is $f_{avg} = \frac{1}{\text{Area}(R)} \cdot \iint_R f(x,y)\,dA$.

(In Calc 1: $f_{avg} = \frac{1}{b-a} \int_a^b f(x)\,dx$)

③ Integral Properties:

f, g are continuous functions on the rectangle R.

$$\iint_R f(x,y) \pm g(x,y) \, dA = \iint_R f(x,y) \, dA \pm \iint_R g(x,y) \, dA$$

if c is scalar

$$\iint_R c\, f(x,y) \, dA = c \cdot \iint_R f(x,y) \, dA$$