11.8 triple integrals in Cylindrical & Spherical Coords.

In $\mathbb{R}^2$ Polar (r, θ) coords.

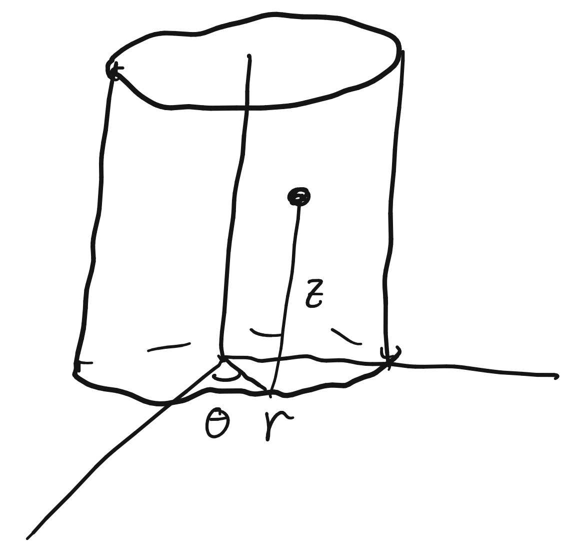
Can do similar ideas in 3D.

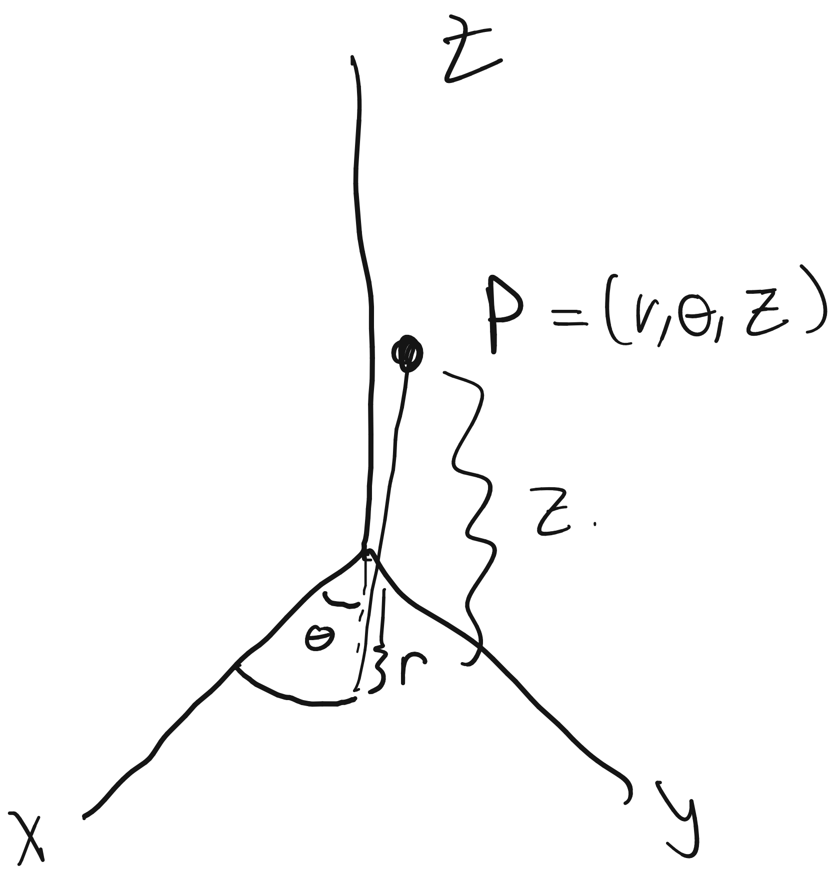
① Cylindrical Coords:

$(\underbrace{r, \theta}, z)$

Std. polar coords on x-y plane

z ↑ height.





Useful when surface/solid has "circular symmetry"

Converting:

Cart. $(x,y,z) \longrightarrow$ Cyl. (r,θ,z) coords

$r^2 = x^2+y^2$

$\tan\theta = y/x$

$z = z.$

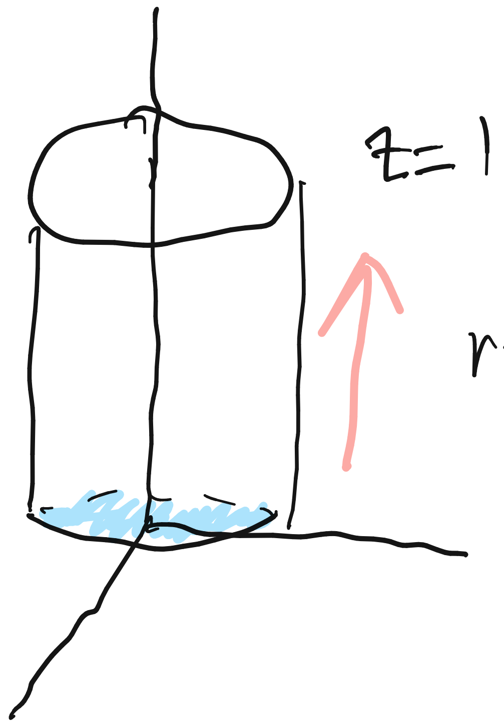
from Cyl. (r,θ,z) coords to Cart (x,y,z)

$x = r\cos\theta, \quad y = r\sin\theta, \quad z = z.$

Geometry ① what does the solid

$0 \le r \le 2, \quad 0 \le z \le 1, \quad 0 \le \theta \le 2\pi$ look like?

Solid cylinder of radius 2, height 1



$z = 1$

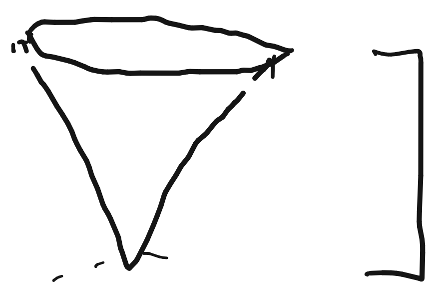
radius $= 2$.

(Solid)

(2) What about the solid $r \le z \le 1$,

$0 \le \theta \le 2\pi$

$0 \le r \le 1$.



ht $= 1$,

radius $= 1$.

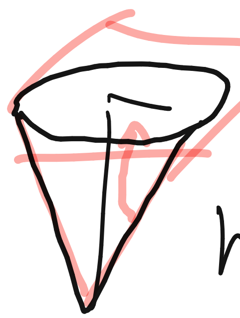
Cone!

Let's do integrals:

$$\boxed{dV = r\,dr\,d\theta\,dz}$$ or permutation of the d-terms.

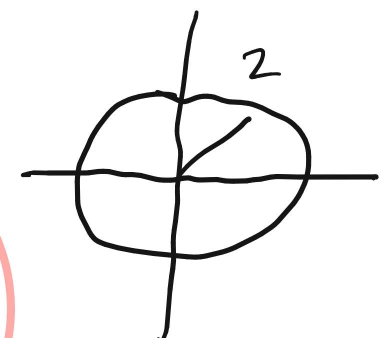
$dA \cdot dz$

Ex Cone $z=\sqrt{x^2+y^2}$ bounded by $z=2$ & x-y plane.



$z=2$

ht $=2$, radius $=2$.



In Cartes. coords:

$$\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{\sqrt{x^2+y^2}}^{2} 1 \, dz\,dy\,dx$$

In Cyl. coords: $r \le z \le 2$

$0 \le \theta \le 2\pi$

$0 \le r \le 2$.

$$\int_0^{2\pi} \int_0^{2} \int_r^{2} \overbrace{r\,dz\,dr\,d\theta}^{dV}$$

$$= \int_0^{2\pi} \int_0^{2} rz \Big|_{z=r}^{z=2} dr\,d\theta = \int_0^{2\pi} \int_0^{2} 2r - r^2 \, dr\,d\theta$$

$$= \int_0^{2\pi} \underbrace{r^2 - \frac{r^3}{3} \Big|_{r=0}^{r=2}} \, d\theta = \int_0^{2\pi} \frac{4}{3} \, d\theta = \boxed{\frac{8\pi}{3}}$$

$$\left(4 - \frac{8}{3}\right) - 0$$

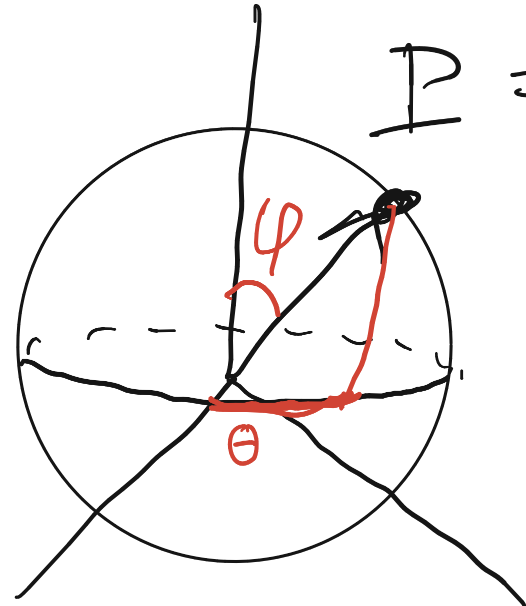
$$\frac{12}{3} - \frac{8}{3} = \frac{4}{3}$$

Spherical Coords

$\phi = \varphi$ "phi"

(ρ, θ, φ)

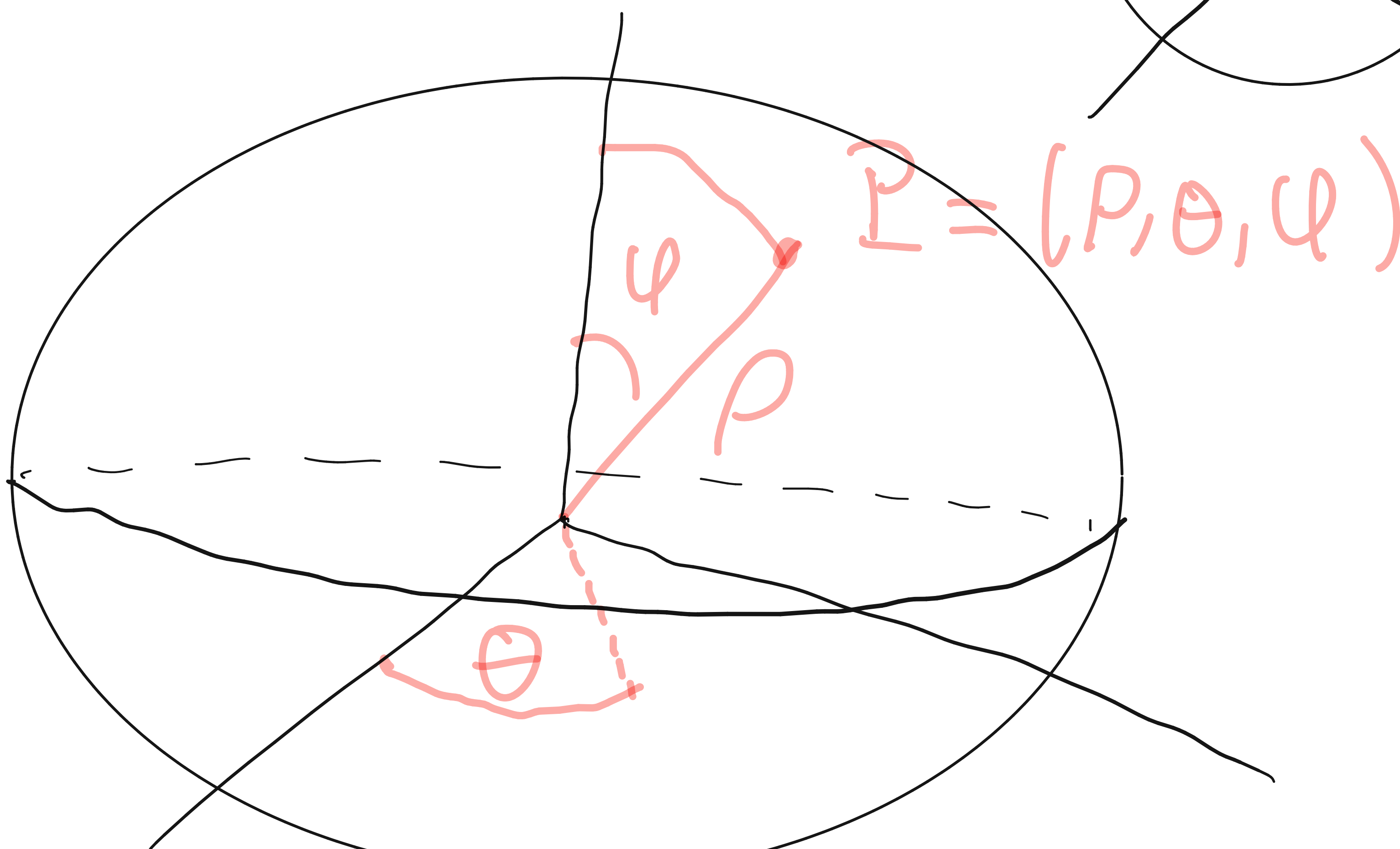
"rho"



ρ tells you dist. from origin

θ: angle you make w/ + x-axis

φ: angle of descent from +z axis.



$0 \le \varphi \le \pi$

from Cart → Spherical:

$$\rho^2 = x^2 + y^2 + z^2$$

$$\tan\theta = \frac{y}{x}$$

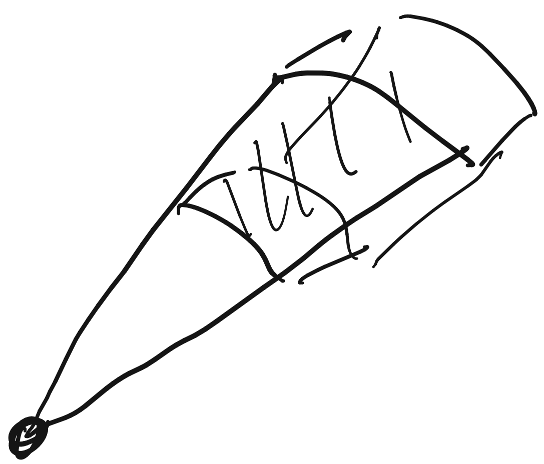
$$z = \rho \cdot \cos(\varphi) \quad \text{or} \quad \cos\varphi = \frac{z}{\rho}$$

from Sph to Cart:

$$x = \rho \sin\varphi \cos\theta$$

$$y = \rho \sin\varphi \sin\theta$$

$$z = \rho \cos\theta.$$

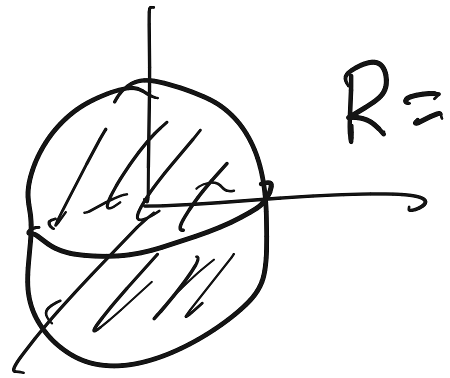


$$\boxed{dV = \rho^2 \sin\varphi \, d\rho \, d\theta \, d\varphi}$$

(or some permutation of the d-terms.)

What does the ~~Surface~~ Solid of $\rho \le 1$ look like?

Solid ball of radius 1 ctd @ origin.



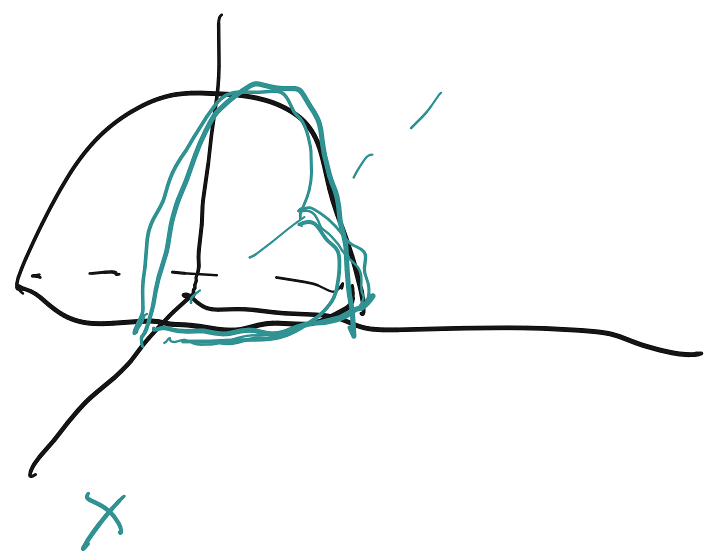
$R = 1$

π

What about solid $0 \le \varphi \le \frac{\pi}{2}$, $0 \le \rho \le 1$, $0 \le \theta \le 2\pi$

solid hemisphere of radius 1

top/ N. Hemisph.



Ex $0^2 \le x^2 + y^2 + z^2 \le a^2$

Solid ball of radius a cent. @ origin.

$$\int_{-a}^{a} \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \int_{-\sqrt{a^2-x^2-y^2}}^{\sqrt{a^2-x^2-y^2}} dz\,dy\,dx$$

gross!

In spherical coords:

$0 \le \rho \le a, \quad 0 \le \theta \le 2\pi, \quad 0 \le \varphi \le \pi$

$$\int_0^{2\pi}\int_0^{\pi}\int_0^{a} \rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi} \frac{\rho^3}{3} \sin\varphi \Big|_{\rho=0}^{a} \, d\varphi \, d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi} \frac{a^3}{3} \sin\varphi \, d\varphi \, d\theta$$

$$= \frac{a^3}{3}\int_0^{2\pi}\int_0^{\pi} \sin\varphi \, d\varphi \, d\theta = \frac{a^3}{3}\int_0^{2\pi} (-\cos\varphi)\Big|_{\varphi=0}^{\pi} \, d\theta$$

$$= \frac{a^3}{3} \cdot \int_0^{2\pi} 2 \, d\theta = \boxed{\frac{4\pi}{3} \cdot a^3} \checkmark$$