§9.8: Arc length (& Re-Parameterization)

Motivation ① What is speed of a curve

② How do we measure the length of a curve?

$\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$

&

Velocity of a curve is $\vec{v}(t) = \vec{r}\,'(t)$

Speed $= \|\vec{v}(t)\| = \|\vec{r}\,'(t)\|$

$$= \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}$$

Recall: Distance travelled $= \int_{\text{start}}^{\text{stop}} \text{Speed} \; dt$

arclength

arclength.

$$S = \int_a^b \|\vec{r}\,'(t)\| \, dt = \int_a^b \sqrt{(x')^2 + (y')^2 + (z')^2} \; dt.$$

In Calc 1/2 we saw:

$y = f(x)$ the arclength between $x = a$, $x = b$ is

$$\int_a^b \sqrt{1 + f'(x)^2}\, dx \qquad t = a \quad t = b.$$

Proof Sketch: Parameterize the graph of $y = f(x)$ as

$$\langle t, f(t) \rangle$$

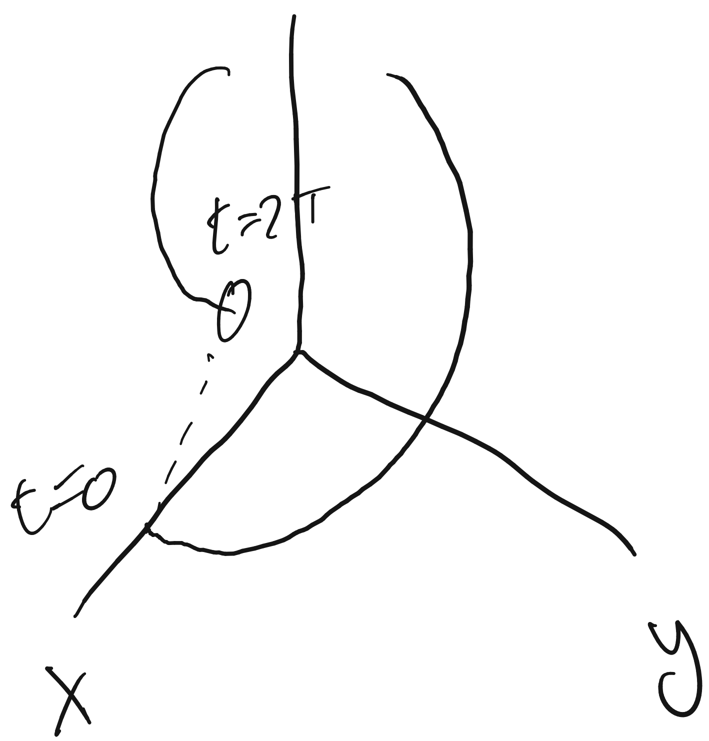
Speed: $\|\vec{r}\,'(t)\| = \sqrt{1^2 + f'(t)^2} = \sqrt{1 + f'(t)^2}$

$$S = \int_a^b \sqrt{1+f'(t)^2}\,dt.$$

Cool!!

Ex Consider the Spiral

$\vec{r}(t) = \langle \cos t, \sin t, t \rangle$ for $0 \le t \le 2\pi$.



Goal: find length of this curve for $0 \le t \le 2\pi$.

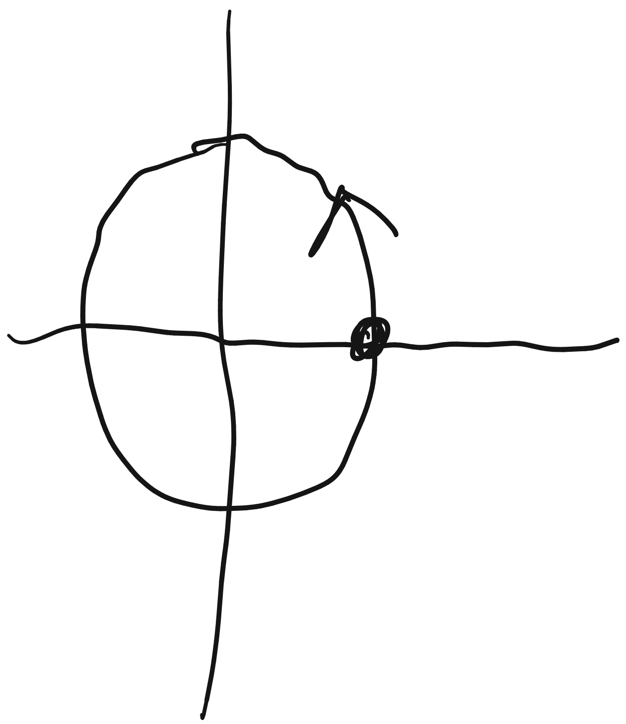
$\vec{r}'(t) = \langle -\sin t, \cos t, 1 \rangle$

$\|\vec{r}'(t)\| = \sqrt{(-\sin t)^2 + (\cos t)^2 + (1)^2}$

$= \sqrt{\underbrace{\sin^2 t + \cos^2 t}_{} + 1} = \sqrt{2}$

So: $S = \int_0^{2\pi} \sqrt{2}\, dt. = \boxed{2\pi\sqrt{2}}$ m

Ex $\langle \cos t, \sin t \rangle$ in $\mathbb{R}^2$.
$= \vec{f}(t)$



$0 \leq t \leq 2\pi$

What does $\vec{f}(2t)$ look like?

$0 \leq t \leq 2\pi$.

double speed $\Rightarrow$ net effect is we traverse the circle 2x.

$r_2(t) = \langle \cos 2t, \sin 2t \rangle \quad 0 \le t \le \pi$

is a "reparameterization" of $\vec{f}(t)$.

traces out same curve,
but slightly differently.

Same curve, written differently.

Def'n let $\vec{r}(t)$ a v-v func.

defnd on $a \le t \le b$.

$$s(t) = \int_a^t \|\vec{r}\,'(\tau)\|\, d\tau \quad \leftarrow \text{(defnd for } a \le t \le b\text{)}.$$

dummy variable

arclength function

$\underline{Ex}$ $\vec{r}(t) = \langle \cos t, \sin t, t \rangle$

$\|\vec{r}\,'(t)\| = \sqrt{2}$.

So $s(t) = \int_0^t \sqrt{2}\, d\tau = t\sqrt{2}$ for this function.

$s = s(t) = t\sqrt{2}$

$s = t\sqrt{2}$

$\Downarrow$

invert & plug back in.

$t = s/\sqrt{2}$

reparameterized function

$$\vec{r}(s) = \left\langle \cos\left(s/\sqrt{2}\right),\ \sin\left(s/\sqrt{2}\right),\ s/\sqrt{2} \right\rangle.$$

this is called arc-length parameterization.

$$\vec{r}(t) = \langle t^2, \frac{8}{3} t^{3/2}, 4t \rangle \quad \text{on } 0 \le t < \infty$$

$$\vec{r}\,'(t) = \langle 2t, 4t^{1/2}, 4 \rangle$$

$$\|\vec{r}\,'(t)\| = \sqrt{4t^2 + 16t + 16} = 2\sqrt{(t+2)^2} = 2(t+2).$$

$$= 2t + 4$$

$$s(t) = \int_0^t (2\tau + 4)\, d\tau = \tau^2 + 4\tau \Big|_{\tau=0}^{t}$$

$$= t^2 + 4t.$$

$S = t^2 + 4t$ or $t^2 + 4t - S = 0$

Solve for t.

By Quad formula! $t = \dfrac{-4 \pm \sqrt{16 + 4S}}{2}$

$$t = \frac{-4 + 2\sqrt{4+S}}{2}$$

$$t = -2 + \sqrt{4+S}$$

$$\vec{r}(t) = \left\langle t^2, \frac{8}{3}t^{3/2}, 4t \right\rangle$$

$$\vec{r}(s) = \left\langle \left(-2+\sqrt{4+s}\right)^2, \frac{8}{3}\left(-2+\sqrt{4+s}\right)^{3/2}, 4\left(-2+\sqrt{4+s}\right) \right\rangle$$

Summary of Steps:

① Compute $\vec{r}\,'(t)$, $\|\vec{r}\,'(t)\|$.

(ie find $s^{-1}(t)$ inverse function)

② Compute $s(t)$, Set $s = s(t)$ Solve for t.

③ Plug in Expression found in ② back in to $\vec{r}(t)$ to get $\vec{r}(s)$.