§12.1 Vector fields (see also §6.1 in Open Stax)

AC

OS

Motivation ① Pure math:

① $f(x)=y$ Single variable functions — 1 input, 1 output

② $f(x_1, \dots, x_n)$ Scalar-valued functions — >1 input, 1 output.

③ $\vec{f}(t)$ Param. functions — 1 input, >1 output.

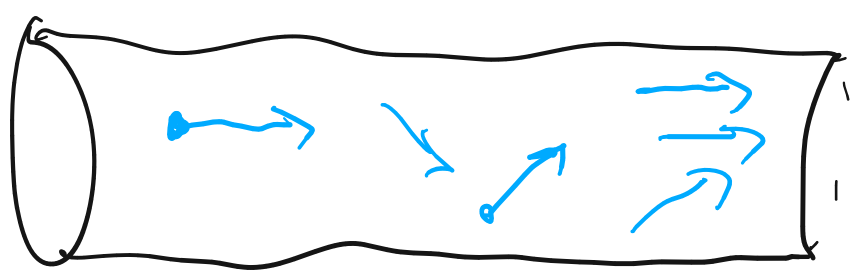
vector-valued funcs.

4 $\vec{f}(x,y,z)$ vector field

>1 inp.
>1 output.

2 Physical Ex:

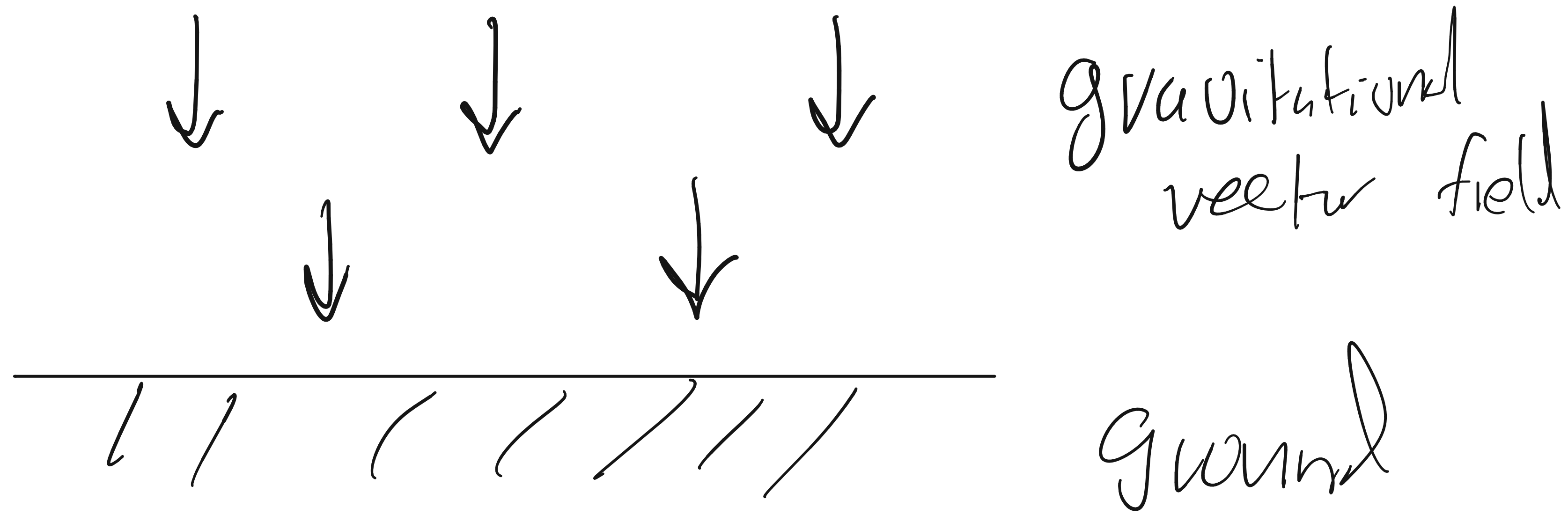
A Consider a fluid flowing through a pipe



This is a Vector field

@ each pos'n (x,y,z) the fluid has a specific velocity vector.

B Gravity:



Def'n: a vector field is a function

$\vec{F}(x,y,z)$ that takes in points as inputs

spits out vectors as outputs.

Notation: $\vec{F}(x,y) = F_1(x,y)\,\hat{\imath} + F_2(x,y)\,\hat{\jmath}$

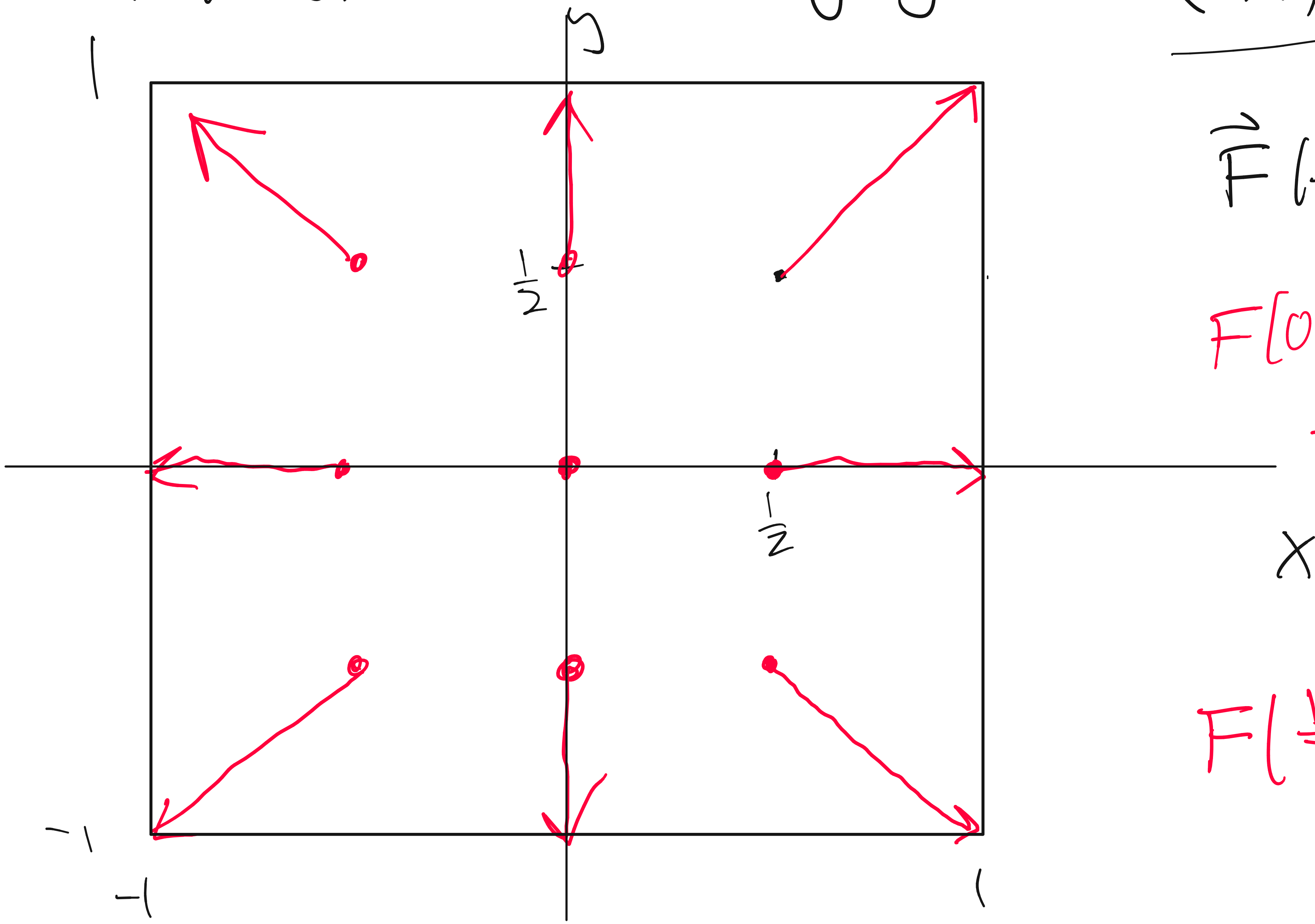
$\vec{F}(x,y,z) = F_1\,\hat{\imath} + F_2\,\hat{\jmath} + F_3\,\hat{k}$ $\qquad F_i = F_i(x,y,z)$

the F_i = "Component functions"

from radiation.

"radial vector field"

Ex $\vec{F}(x,y) = x\hat{i} + y\hat{j} = \langle x, y \rangle$



$\vec{F}(\frac{1}{2},\frac{1}{2}) = \langle \frac{1}{2}, \frac{1}{2} \rangle$

$F(0,0) = \langle 0, 0 \rangle$

$F(\frac{1}{2}, -\frac{1}{2})$

$F(\frac{1}{2}, 0) = \langle \frac{1}{2}, 0 \rangle$

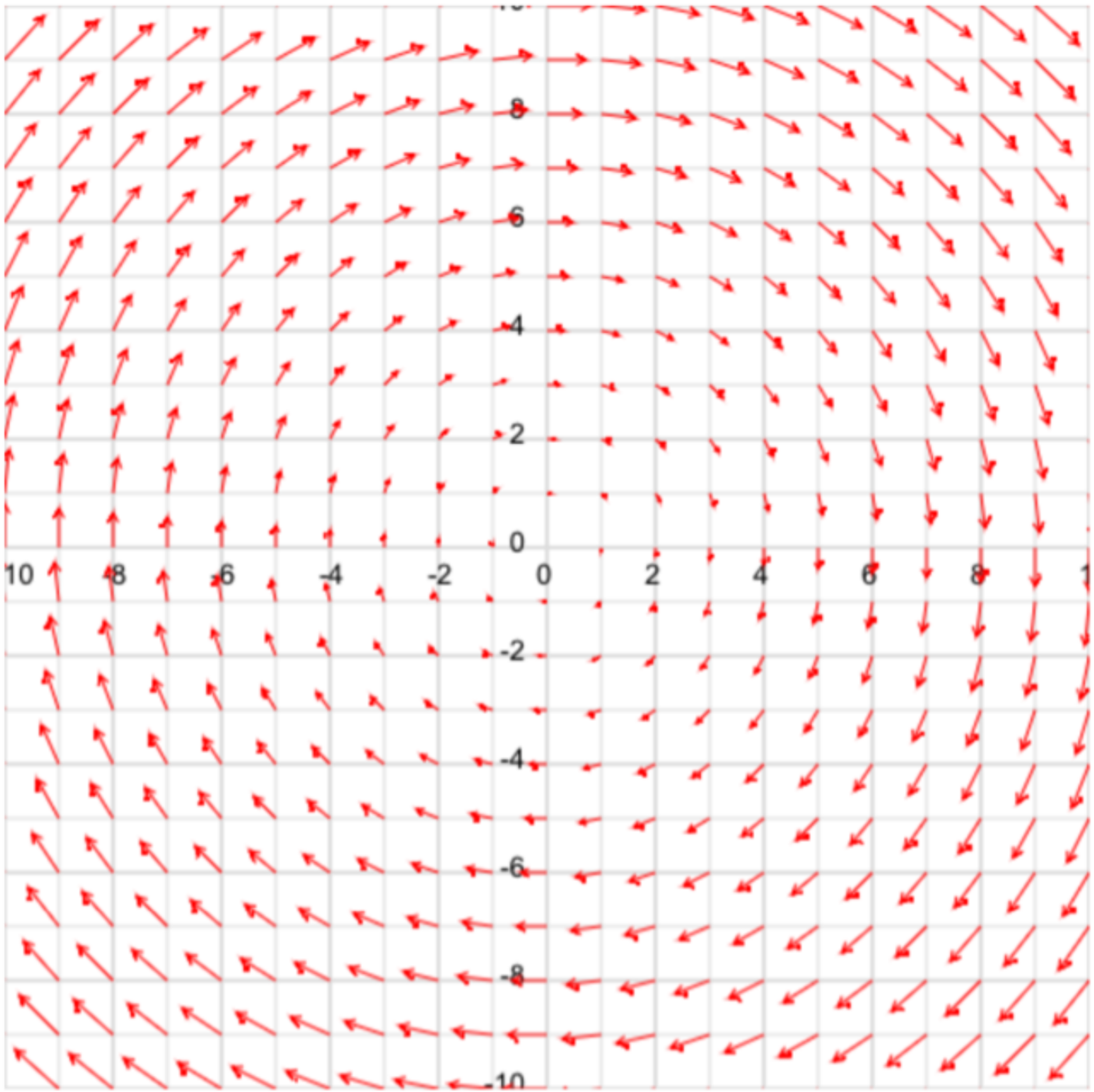
$\boxed{\text{How to plot a V-field}}$

Repeat following process

1. Pick point @ random
2. Compute $\vec{F}$ @ that point
3. plot the resulting vector ~~based~~ @ that point.

<u>Ex</u> $\vec{G}(x,y) = \langle y, -x \rangle$.

1. Plot $\vec{G}(x,y)$ on domain $\underset{x}{[-1,1]} \times \underset{y}{[-1,1]}$.



rotational vector field

$\vec{G} = \langle y, -x \rangle$

(b) $G = \langle y, -x \rangle$

Verify that $\vec{F} = \langle x, y \rangle$ is perpendicular to $\vec{G}$ @ every point (x, y).

$\vec{G} = \langle y, -x \rangle$

$$\vec{G} \cdot \vec{F} = \langle y, -x \rangle \cdot \langle x, y \rangle = yx - xy$$

$$= 0$$

So $\vec{G} \perp \vec{F}$. @ every point (x, y).

Ex Gradient vector fields:

$$f(x,y,z) \longmapsto \vec{\nabla} f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle$$

this is a vector field!

Def'n: If $\vec{F}(x,y,z)$ vector field and if we can find a $f(x,y,z)$ such that $\vec{\nabla} f = \vec{F}$ we call $f(x,y,z)$ a Pontential function for $\vec{F}$.

Think: potential function is like an anti-derivative.

Ex ① The function $f(x,y) = \frac{1}{2}x^2 + \frac{1}{2}y^2$ is a pontential function for $\vec{F}(x,y) = \langle x, y \rangle$.

Verify this:

$$\vec{\nabla} f = \langle x, \tfrac{1}{2}\cdot 2y \rangle = \langle x, y \rangle = \vec{F} \checkmark$$

② How do we Construct a potential function?

$\vec{F} = \langle 6xy, \boxed{3x^2 + 9\sqrt{y}} \rangle$. $\vec{F}$ is a gradient vector field.

find a $f(x,y)$ st $\underline{\vec{\nabla} f = \vec{F}}$.

$\boxed{\dfrac{\partial f}{\partial x} = 6xy}$, and $\partial/\partial y$

$\dfrac{\partial f}{\partial y} = 3x^2 + 9\sqrt{y}$

① Make a good solid guess.

$f = 3x^2 y + g(y)$

b/c since g depends only on y, $\dfrac{\partial g}{\partial x} = 0$

$$\frac{\partial f}{\partial y} = \cancel{3x^2} + g'(y) = \cancel{3x^2} + 9\sqrt{y}$$

F_y

$$\Downarrow$$

$$g'(y) = 9\sqrt{y}$$

$\Downarrow$ anti-derivative

$$\Rightarrow \underline{g(y)} = 9 \cdot \frac{2}{3} \cdot y^{3/2} + C = 6y^{3/2} + C$$

$$\Rightarrow \boxed{f(x,y) = 3x^2y + 6y^{3/2} + C}$$

This process helps us go between

Scalar functions

$f(x,y)$ & vector fields

$\vec{F}(x,y)$.