12.5 Path-Independent V-fields & The Fundamental Theorem of Calculus for line integrals

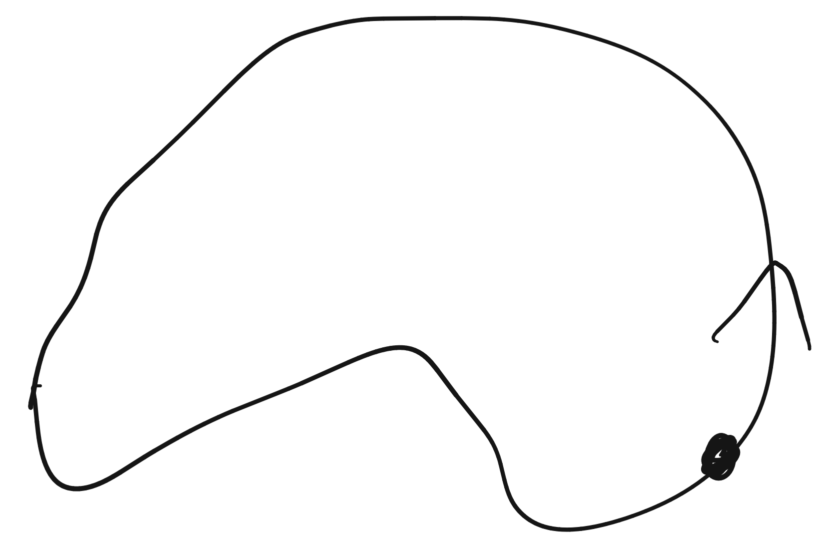
(cf OS 6.3)

Last time: $\vec{F}$: V field

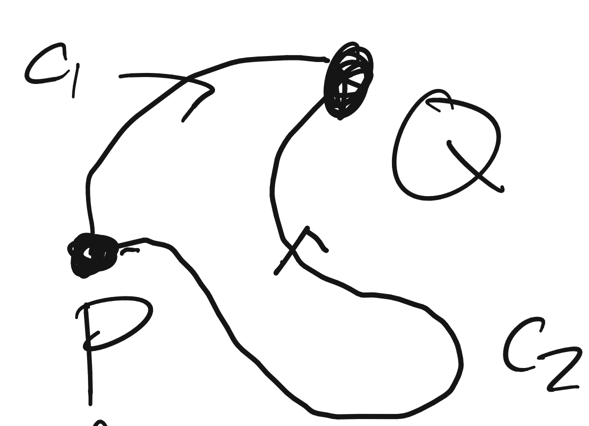
C : Oriented curve

$\vec{r}(t)$ Parameterization of C $\quad a \le t \le b$

$$\int_C \vec{F} \cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt$$



Path-Indep. Vector field: (PI)



$\vec{F}$ v.f. $\quad C_1, C_2$ oriented curves w/ same endpoints

$\vec{F}$ is P.I if $\displaystyle\int_{C_1} \vec{F}\cdot d\vec{r} = \int_{C_2} \vec{F}\cdot d\vec{r}$

Big Qs:

1. How do we tell if a V.f. $\vec{F}$ is P.I.?

2. If $\vec{F}$ is P.I can we simplify the calculation of $\int_C \vec{F}\cdot d\vec{r}$ to not depend on any parameterizations?

3. How do gradients play into all of this?

Theorem: (Fundamental Theorem of Calculus for Line Integrals)

Let f be a scalar function,

C is an oriented curve from pt P to pt Q.

Then:

$$\boxed{\int_C \nabla \vec{f} \cdot d\vec{r} = f(Q) - f(P)}$$

Ex $f(x,y) = 3xy^2 - \sin x + e^y$

C oriented half circle from $(-1,0)$ to $(1,0)$

P Q

Compute $\int_C \vec{\nabla} f \cdot d\vec{r}$

$\nabla f = \langle 3y^2 - \cos x, 6xy + e^y \rangle$

$\vec{r}(t) = \langle -\cos t, \sin t \rangle \quad 0 \le t \le \pi$

Ew!

Using FTC - LI: $\quad Q = (1,0) \quad P = (-1,0)$

$$\int_C \vec{\nabla} f \cdot d\vec{r} = f(Q) - f(P) \quad \text{"Stop - Start."}$$

$$= f(1,0) - f(-1,0)$$

$$f(x,y) = 3xy^2 - \sin x + e^y$$

$$\Rightarrow \int_C \vec{\nabla} f \cdot d\vec{r} = (\cancel{3 \cdot 1 \cdot 0^2}^{\,0} - \sin(1) \cancel{+ e^0}) - (\cancel{3 \cdot (-1) \cdot 0^2}^{\,0} - \sin(-1) \cancel{+ e^0})$$

$$= \sin(-1) - \sin(1).$$

<u>Theorem:</u> $\boxed{\text{If } \vec{F} \text{ is P.I. then } \vec{F} = \vec{\nabla} f \text{ some scalar function } f.}$

i.e. Every path-indep. v.f <u>is</u> a gradient v.f.!

<u>Q</u>: What if C is a <u>closed curve</u>?

① Case 1: F is path-indep. (ie. F <u>is</u> a gradient).

$$P=Q \Rightarrow \oint_C \vec{F}\cdot d\vec{r} = \oint_C \vec{\nabla} f \cdot d\vec{r} = f(Q) - f(P)$$
$$= f(Q) - f(Q) = 0$$

__Result:__ If $\vec{F}$ is path-independent and C is a closed curve, then

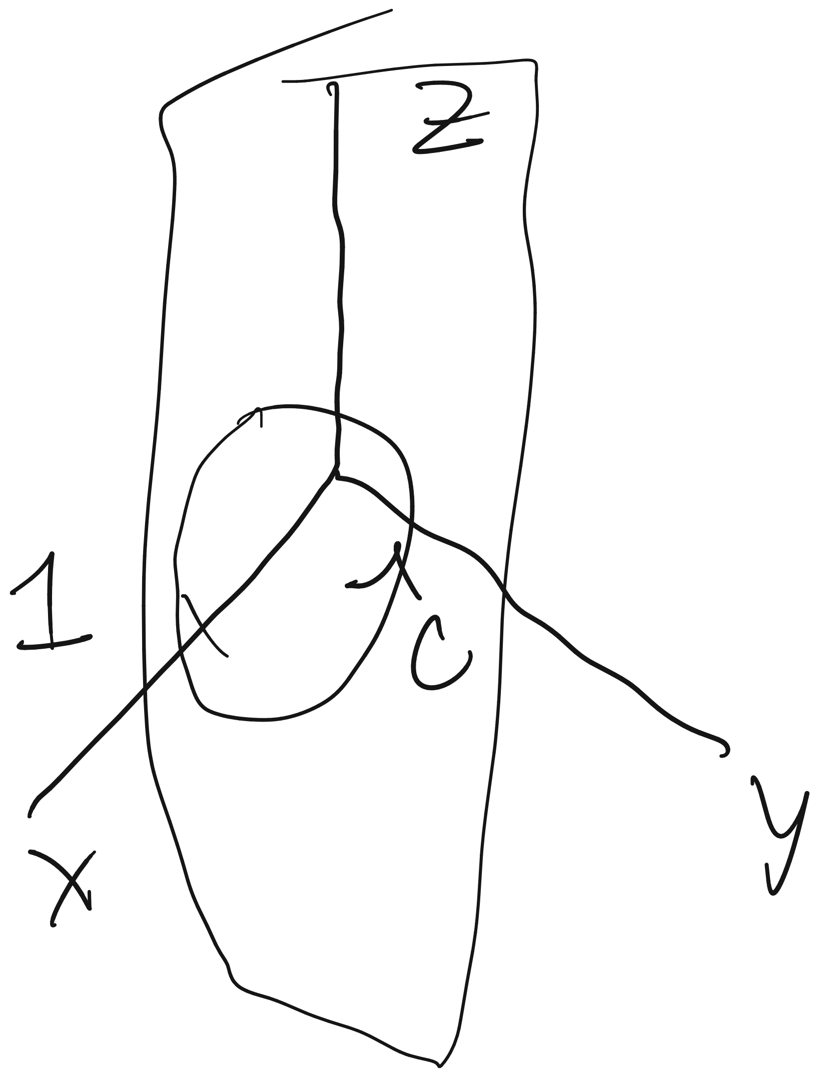
$$\oint_C \vec{F}\cdot d\vec{r} = 0.$$

__Side Effect:__ If C is a closed curve and $\oint_C \vec{F}\cdot d\vec{r} \neq 0$, then $\vec{F}$ is __not__ path independent!

Ex $\vec{F} = \langle x, z, 3y \rangle$

C is param. by $\vec{r}(t) = \langle 1, \sin t, \cos t \rangle$

$0 \le t \le 2\pi$.



$$\oint_C \vec{F} \cdot d\vec{r} = -2\pi \neq 0$$

$\Rightarrow \vec{F}$ is <u>not</u> p. I.

This is <u>hard</u>.

Theorem: (2D Path-independence check)

$\vec{F} = \langle F_1, F_2 \rangle$

F is not P.I. $\boxed{\text{if } \frac{\partial}{\partial x} F_2 \neq \frac{\partial}{\partial y} F_1}$

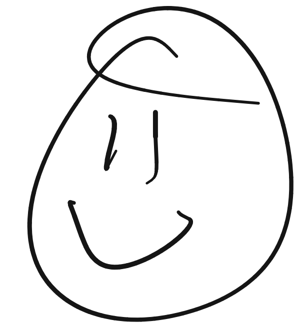
Proof: Suppose F is P.I., so $\vec{F} = \vec{\nabla} f$ some scalar function f.

$$F_1 = f_x, \quad F_2 = f_y$$

$$(F_1)_y = f_{xy} \quad , \quad (F_2)_x = f_{yx}$$

But by mixed second partials theorem

$$f_{xy} = f_{yx} \Rightarrow (F_1)_y = (F_2)_x .$$



Ex $F = \langle 2y, 3x \rangle$

Q: Is F P.I. or not?

A: **No!** $\frac{\partial F_1}{\partial y} = 2$, but $\frac{\partial F_2}{\partial x} = 3$

$$2 \neq 3.$$