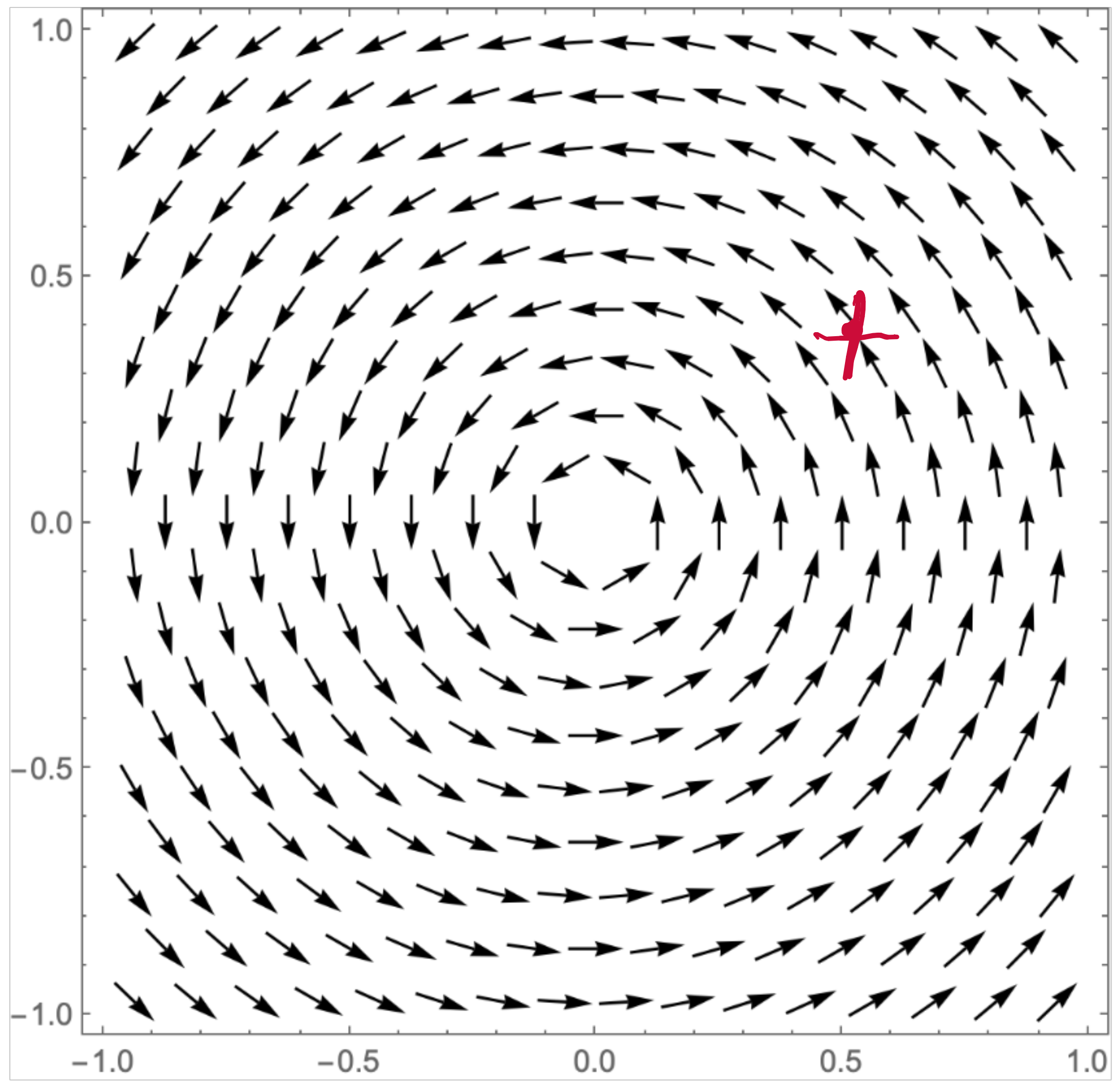
AC 12.7 / OS 6.5: Curl of Vector fields
(and OS 6.3)

Curl measures "rotational force" of a v. field.

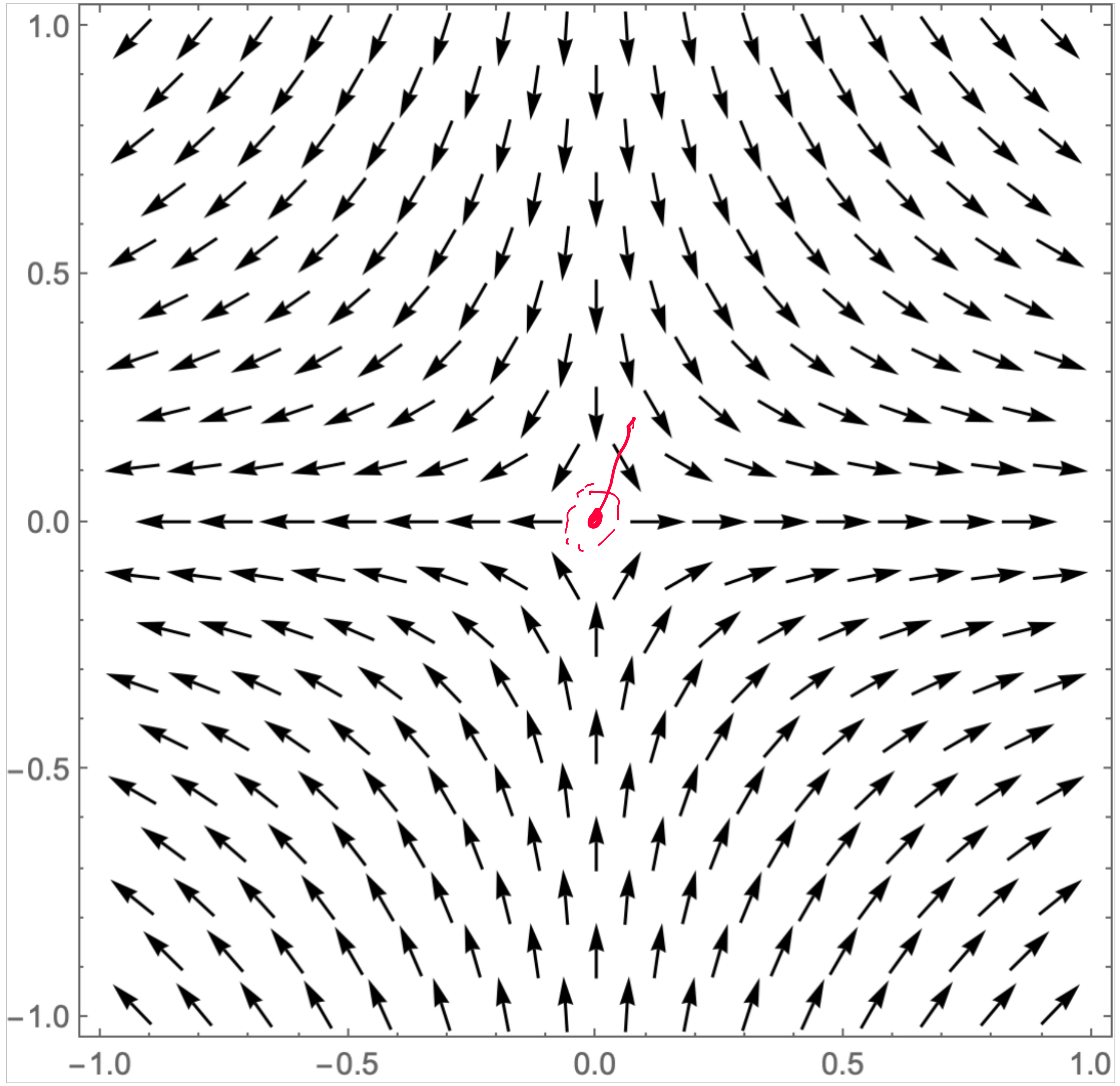
Ex $F = \langle y, -x \rangle$



$\cdot p$

$\mathrm{Curl}[F](p)$

not as "spinny" as above.



dir = axis of rot'n
magnitude = "how 'spinny' it is".

In 2D: $\vec{F} = \langle P(x,y), Q(x,y) \rangle$

$$\text{Curl}(\vec{F}) = (Q_x - P_y)\,\hat{k}$$

all funcs of (x,y,z)

In 3D $\vec{F} = \langle P, Q, R \rangle$

$$\vec{\nabla} = \langle \partial/\partial x, \tfrac{\partial}{\partial y}, \partial/\partial z \rangle.$$

$$\text{Curl}(\vec{F}) = \vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \partial/\partial y & \partial/\partial z \\ P & Q & R \end{vmatrix}$$

Ex ① $\vec{F} = \langle y, 0 \rangle$

$$\text{Curl}(\vec{F}) = (0-1)\hat{k} = -\hat{k}$$

$$= \langle 0, 0, -1 \rangle.$$

Something to note: $\vec{F}$ is not path independent.

② $\vec{F} = \langle \underset{P}{\underset{\uparrow}{x-y}}, \underset{Q}{\underset{\uparrow}{y+2z}}, \underset{R}{\underset{\uparrow}{x^2}} \rangle$

$$\text{Curl}(F) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = \hat{i}(R_y - Q_z) - \hat{j}(R_x - P_z) + \hat{k}(Q_x - P_y)$$

$R_y = 0 \qquad R_x = 2x$

$Q_z = 2 \qquad P_z = 0$

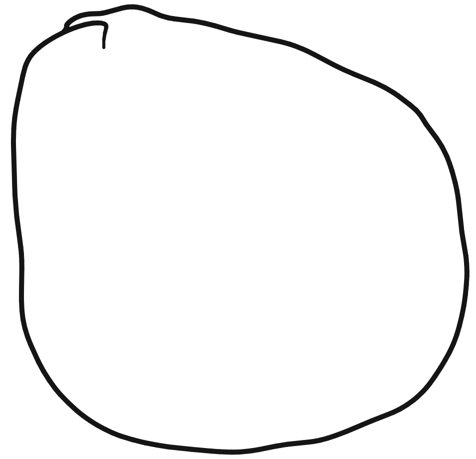
$Q_x = 0, \quad P_y = -1$

$$\text{Curl}(F) = \hat{i}(-2) - \hat{j}(2x) + \hat{k}(1)$$

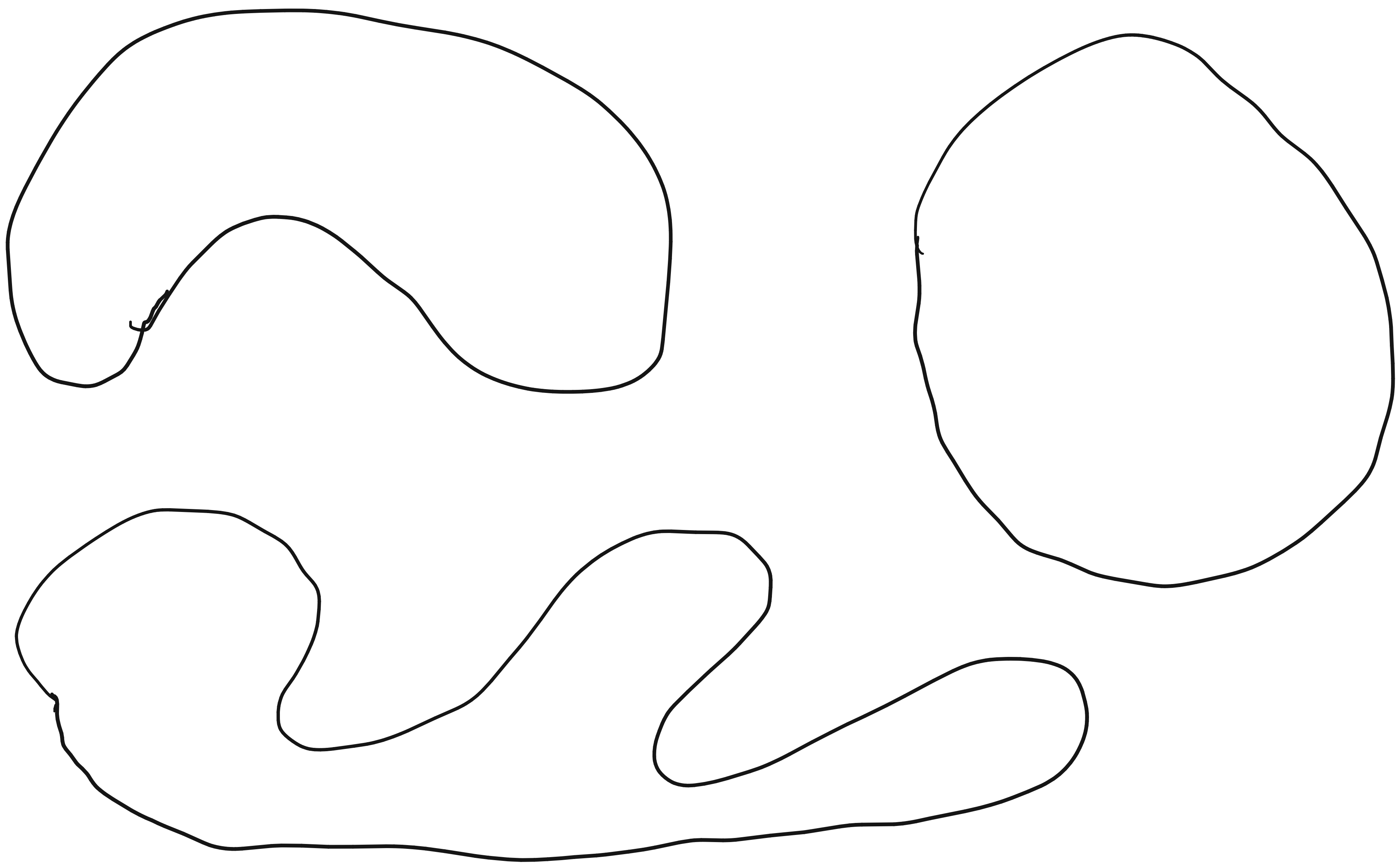
$$= \langle -2, -2x, 1 \rangle \neq \vec{0}$$

Claim: this $\vec{F}$ is also not P.I.

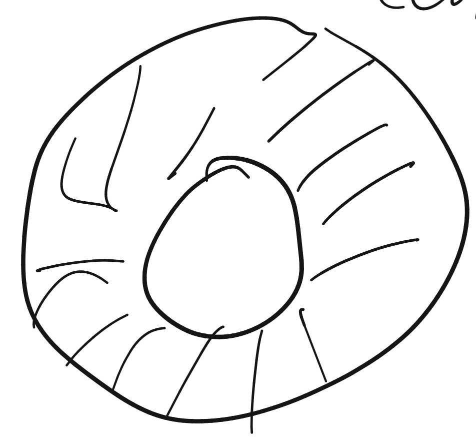
Theorem: A 2D or 3D V. field $\vec{F}$ (defined on a closed, simply connected region) is P.I if and only if $\text{Curl}(F) = \vec{0}$.

Closed = 

Simply connected = connected and "no holes".



not simply connected.



Takeaway: $\boxed{\text{Curl}(\vec{F}) = \vec{0}} \iff \vec{F}$ is P.I.

$\Updownarrow$

$\exists$ scalar function f such that $\vec{\nabla} f = \vec{F}$

Curl test:

If $\text{Curl}(\vec{F}) \neq \vec{0}$ then $\vec{F}$ is not P.I.

Ex Use Curl test to determine if

$\vec{F} = \langle \overset{P}{yz}, \overset{Q}{xz}, \overset{R}{xy} \rangle$ is P.I., if it is,

find a f st $\vec{\nabla} f = \vec{F}$

① Compute curl: $\quad \partial_x = \dfrac{\partial}{\partial x}$

$$\text{Curl}(\vec{F}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ P & Q & R \end{vmatrix} = \left[\hat{i}\,\big(\cancel{R_y - Q_z}\big)^{\to 0} - \hat{j}\,\big(\cancel{R_x - P_z}\big)^{\to 0} + \hat{k}\,\big(\cancel{Q_x - P_y}\big)^{\to 0} \right]$$

$$= \vec{0}.$$

$R_y = x \qquad R_x = y \qquad Q_x = z$

$Q_z = x \qquad P_z = y \qquad P_y = z$

$\text{Curl}(\vec{F}) = \vec{0}$ so by the theorem, we must have that $\vec{F}$ is P.I.!

② want to find a potential function $f(x,y,z)$ st

$\vec{\nabla} f = \vec{F}$

$F = \langle yz, xz, yx \rangle$

guess: $f(x,y,z) = ?$

$f_x = yz$

$f_y = xz$

$f_z = yx$

$$\boxed{f(x,y,z) = xyz + g(y,z)}$$

↓ look @ y-partial deriv.

$$xz = \underbrace{xz + g_y(y,z)}$$ $\Rightarrow$ $g_y(y,z) = 0$ for all y, z.

$\uparrow$ want

What guess gives me.

$\Rightarrow$ g does <u>not</u> depend on y.

$\Rightarrow$ $g(y,z) = h(z)$ some func h

<u>New guess!</u>

$$f(x,y,z) = \underbrace{xyz + h(z)}$$

$f_z = xy = xy + h'(z) \Rightarrow h'(z) = 0$ for all z

$\Rightarrow h(z) = C$

Final Sol'n: $f(x,y,z) = xyz + C$ Some Constant # C.

<u>Why?</u> $\displaystyle\int_C F \cdot d\vec{r} = f(1,1,1) - f(0,0,0)$
$= 1 - 0 = \underline{1}$

C is any path from $(0,0,0)$ to $(1,1,1)$

Method for Computing line integrals in general:

Given: $\vec{F}$ a vector field

C: an oriented curve.

① Determine if F is P.I. or not.

(use $\boxed{\text{Curl test}}$ or Circulation test.)

$\oint_D \vec{F} \cdot d\vec{r} = 0$ for all closed D?

(2) (A) If $\vec{F}$ is PI, find a pontential func.

f st. $\vec{\nabla} f = \vec{F}$.

↳ use FTC for LIs to compute

$$\int_C \vec{F} \cdot d\vec{r} = \int_C \vec{\nabla} f \cdot d\vec{r} = f(B) - f(A).$$

(B) If F is not PI, use a parameterization

$\vec{r}(t)$ for C to compute the L.I.