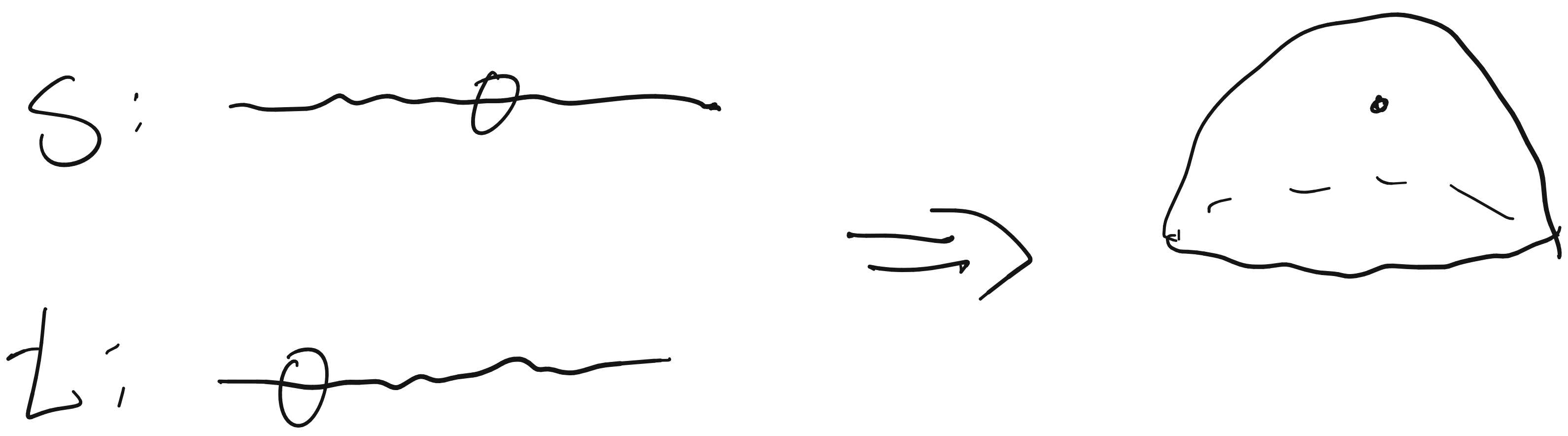
AC 11.6 / OS 6.6 | Parametric Surfaces & Surface area

<u>Idea</u>: So far we can integrate V-fields over 1-dimensional spaces

<u>Goal</u>: Integrate V-fields over 2-D spaces, ie Surfaces.

Today, we'll build up to this, but we need a new tool first.

Idea of a parametric surface is the following



2 parameters sweeps out a 2D surface.

Def'n a <u>parametrization of a surface</u> is a v-v function

$$\vec{r}(s,t) = \langle x(s,t), y(s,t), z(s,t) \rangle = x(s,t)\hat{i} + y(s,t)\hat{j} + z(s,t)\hat{k}$$

graph of $\vec{r}(s,t)$ is called a <u>surface</u> or a <u>parametric surface</u>.

<u>Ex</u> ① $z = f(x,y)$ is a "height function"

If we want to param. the graph of $f(x,y)$ do the following:

$$\boxed{\vec{r}(s,t) = \langle s, t, f(s,t) \rangle}$$

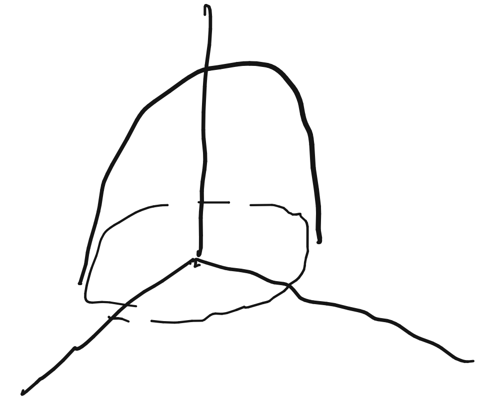
$x(s,t) = s$

$y(s,t) = t$

$z(s,t) =$

Ex ② hemisphere centered @ origin w/ Radius $R > 0$

$$z = \sqrt{R^2 - x^2 - y^2} \Leftrightarrow x^2 + y^2 + z^2 = R^2$$



$$\boxed{\vec{r}(s,t) = \langle s, t, \sqrt{R^2 - s^2 - t^2} \rangle}$$

$0 \le s^2 + t^2 \le 1.$

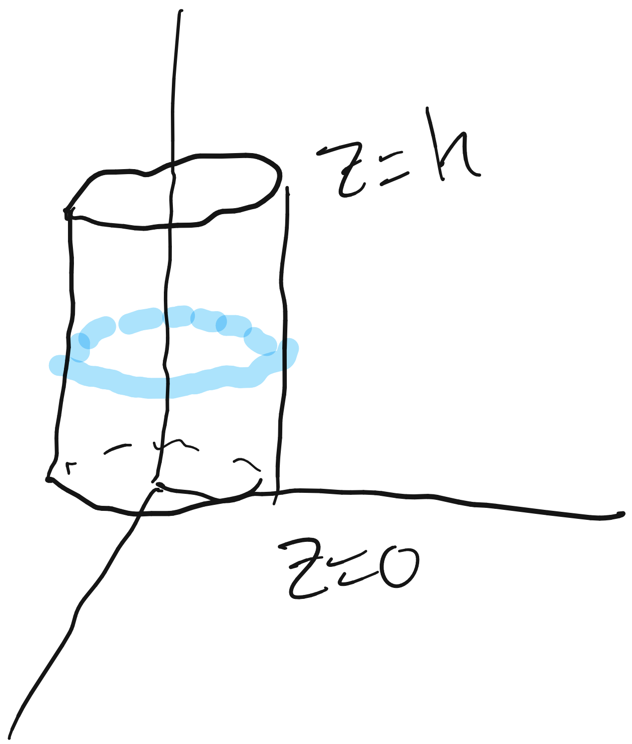
s plays role of x

t plays role of y

NB: Some sources use us, vs rather than ss ts.

Ex ③ Cylinder of radius a, height h

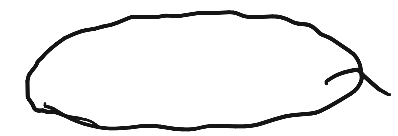
w/ bottom on xy-plane.



$x(s,t) = a \cdot \cos(s)$

$y(s,t) = a \cdot \sin(s)$

$z(s,t) = t$



$\langle a \cdot \cos(s), a \cdot \sin(s), t \rangle.$

$0 \leq s \leq 2\pi$

$0 \leq t \leq h$

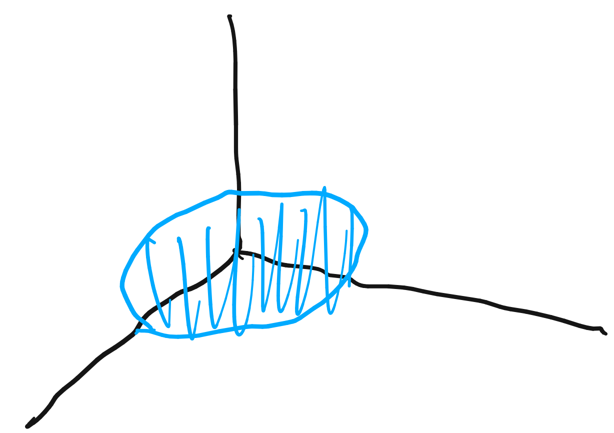
Ex ④: **Plane:** $x+y+z=1$

rearrange: $z = 1-x-y$

Param: $\langle s, t, 1-s-t \rangle$

$0 \le s, t \le 1.$

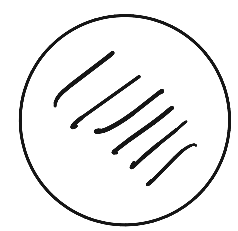
Ex ⑤ **Disk** $x^2+y^2 \le 1$ $z=0$.



$z(s,t) = 0.$

$x(s,t) = s \cdot \cos(t)$

$y(s,t) = s \cdot \sin(t)$



$0 \le s \le 1$ ← playing role of r

$0 \le t \le 2\pi$ ← playing role of θ

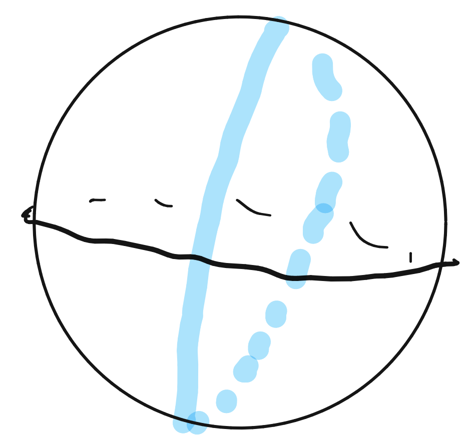
Ex ⑥ Sphere of radius R centered @ origin.

Idea: Steal from Spherical coords.

<u>Claim:</u> $\vec{r}(s,t) = \langle R\cos s\sin t, R\sin s\sin t, R\cos t\rangle$

$0 \leq s \leq 2\pi \leftarrow \theta$

$0 \leq t \leq \pi \leftarrow \varphi$



Application Surface area!

S is a surface parameterized by

$\vec{r}(s,t)$ where domain$(\vec{r}) = D \subseteq \mathbb{R}^2$

then $\text{Area}(S) = \iint_D \left\| \frac{\partial \vec{r}}{\partial s} \times \frac{\partial \vec{r}}{\partial t} \right\| dA$

$$= \iint_D \| \vec{r}_s \times \vec{r}_t \| \, dA$$

Ex C is cylinder we did before.

$$\vec{r}(s,t) = \langle a \cdot \cos(s),\ a \cdot \sin(s),\ t \rangle.$$

$0 \le s \le 2\pi$
$0 \le t \le h.$
← Domain D.

1. $\vec{r}_s$, $\vec{r}_t$

$$\vec{r}_s = \langle -a\sin s,\ a\cos s,\ 0 \rangle$$

$$\vec{r}_t = \langle 0,\ 0,\ 1 \rangle$$

② $$\vec{r}_s \times \vec{r}_t = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -a\sin s & a\cos s & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= a\cos(s)\,\hat{i} + a\sin(s)\,\hat{j}$$

$$= \langle a\cos(s), a\sin(s), 0 \rangle.$$

③ $$\|\vec{r}_s \times \vec{r}_t\| = \sqrt{a^2\cos^2 s + a^2\sin^2 s + 0^2} = a$$

④ $$\text{Area} = \int_0^h \int_0^{2\pi} a \, ds\, dt = \boxed{2\pi a h.}$$

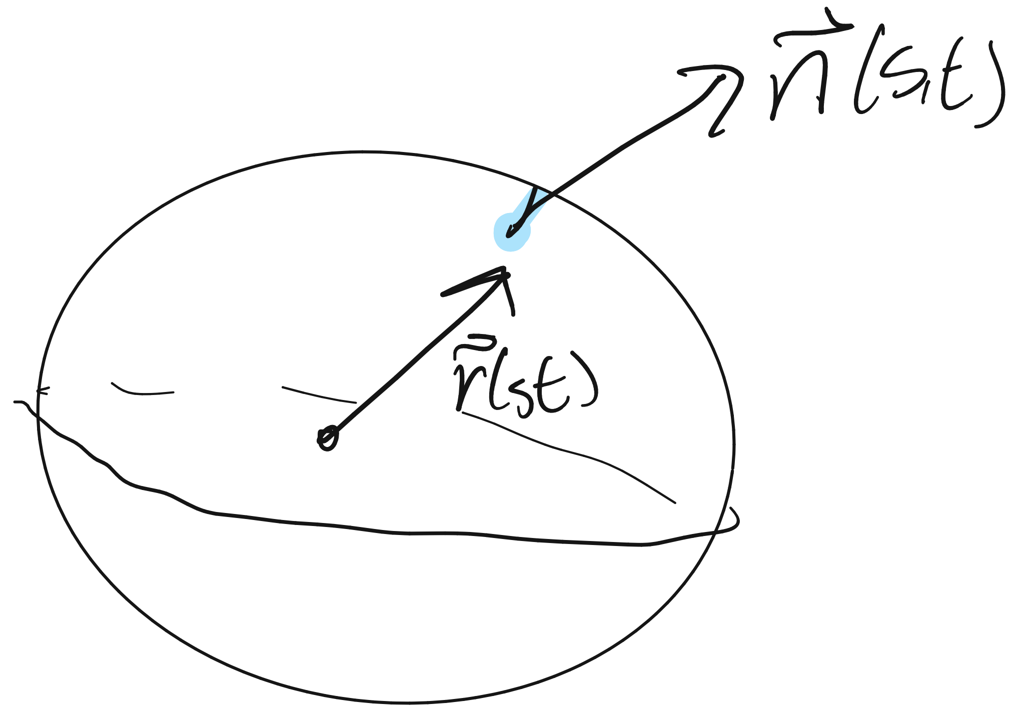
Notation: the surface normal vector is

$$\boxed{\vec{n} = \vec{r}_s \times \vec{r}_t}$$

1^{st} compute $\vec{r}(s,t)$, $\vec{n}(s,t)$

$\vec{n}(s,t)$

Picture



$$\hat{n} = \frac{\vec{n}}{\|\vec{n}\|}.$$ unit normal vector