AC 12.9 || OS 6.6 : Flux/Surface Integrals

Last time: we saw how to parameterize surfaces using V-V funcs dependent on two parameters

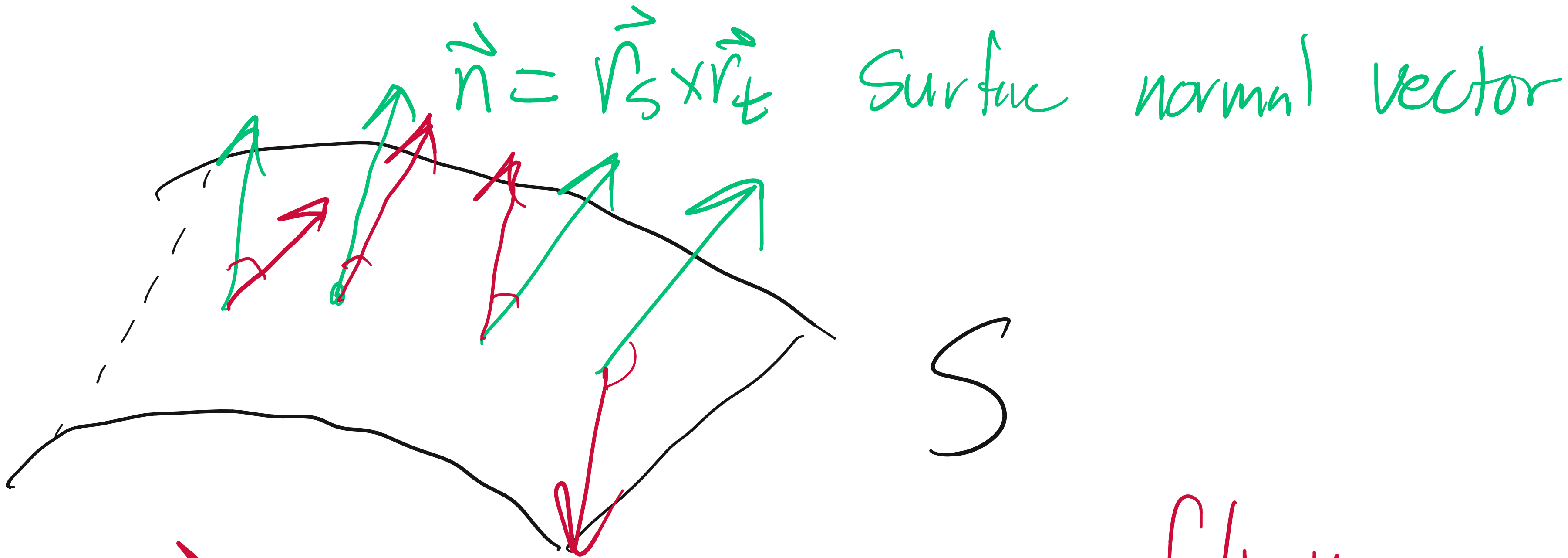
$\vec{r}(s,t)$ parameterize some surface S

We also saw how to compute surface area:

$$A(S) = \iint_D \|\vec{r}_s \times \vec{r}_t\| \, dA$$

$D =$ domain of $\vec{r}(s,t)$.

Picture:



$\vec{F} = \vec{F}(x,y,z)$

flux or Surface integral

$$\text{flux} = \sum_{\substack{\text{Parts}\\ \text{of}\\ \text{Surface}}} \vec{F} \cdot \vec{n}\ |\text{Area}| = \boxed{\iint_S \vec{F} \cdot \vec{n}\, dA}$$

$\vec{n} = \vec{r}_s \times \vec{r}_t$

$\vec{r}(s,t)$ is a Param. of my Surf. S.

Intuitively, flux = # arrows out − # arrows in

very roughly speaking

<u>Ex</u> S will be plane $\boxed{\begin{array}{l} \underline{x+y+z=1}, \\ 0 \le x, y \le 1. \end{array}}$

$$\vec{F} = \langle -y, x, 0 \rangle$$

<u>Goal</u>: Compute $\iint_S \vec{F} \cdot \vec{n} \, dA$

① Parameterize S

$z = 1 - x - y \qquad 0 \le x, y \le 1$

"Standard trick": $\langle s, t, f(s,t) \rangle$

$\vec{r}(s,t) = \langle s, t, 1-s-t \rangle \qquad 0 \le s, t \le 1$

② Compute $\vec{n} = \vec{r}_s \times \vec{r}_t$

$0 \le s \le 1$; $0 \le t \le 1$. $= \mathrm{Dom}(\vec{r})$

$\vec{r}_s = \langle 1, 0, -1 \rangle \qquad \vec{r}_t = \langle 0, 1, -1 \rangle$

$$\vec{r}_s \times \vec{r}_t = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{vmatrix} = \langle 1, 1, 1 \rangle = \vec{n}$$

(3) Compute $\vec{F}(\vec{r}(s,t))$

$$F = \langle -y, x, 0 \rangle$$

$$r(s,t) = \langle \underbrace{s}_{=x}, \overbrace{t}^{=y}, \overbrace{1-s-t}^{z=} \rangle.$$

$$\vec{F}(\vec{r}(s,t)) = \langle -t, s, 0 \rangle$$

(4) $\vec{F}(\vec{r}(s,t)) \cdot \vec{n}$

$$= \langle -t, s, 0 \rangle \cdot \langle 1, 1, 1 \rangle$$

$$= -t + s = \boxed{s - t}$$

this is what we integrate.

⑤ Compute it!

$$\iint_S \vec{F}\cdot\vec{n}\,dA = \int_0^1\int_0^1 (s-t)\,ds\,dt$$

$$= \int_0^1 \frac{1}{2}s^2 - st\Big|_{s=0}^{1}\,dt$$

$$= \int_0^1 \frac{1}{2} - t\,dt$$

$$= \frac{1}{2}t - \frac{1}{2}t^2\Big|_0^1 = \boxed{0}$$

What we found:

$$\iint_S \vec{F} \cdot \vec{n} \, dA = 0.$$

Ex $F = \langle y, x, \underbrace{z + \sin x} \rangle$

S cylinder of radius 2 centered on z-axis w/ $0 \le z \le 3$.

Aside: Some sources use

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_S \vec{F} \cdot \vec{n} \, dA$$

Goal: $\iint_S \vec{F} \cdot \vec{n} \, dA$

1. Parameterize
2. Compute $\vec{n}$
3. Compute dot product
4. Integrate.

(1) $\vec{r}(s,t) = \langle 2\cos(s), 2\sin(s), t \rangle$

$0 \le s \le 2\pi$

$0 \le t \le 3$.

(2) $\vec{n} = \vec{r}_s \times \vec{r}_t$

$$\vec{r}_s = \langle -2\sin s,\ 2\cos s,\ 0 \rangle$$

$$\vec{r}_t = \langle 0, 0, 1 \rangle$$

$$\vec{n} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -2\sin s & 2\cos s & 0 \\ 0 & 0 & 1 \end{vmatrix} = \boxed{\langle 2\cos s,\ 2\sin s,\ 0 \rangle}$$

③ Compute $\vec{F}(\vec{r}(s,t))$, $\vec{F}(\vec{r}(s,t)) \cdot \vec{n}$

$$\vec{F}(\vec{r}(s,t)) = \langle 2\sin s,\ 2\cos s,\ 2+\sin(2\cos s)\rangle$$

$$\vec{F}(\vec{r}) \cdot \vec{n} = \text{'' \quad ''} \cdot \langle 2\cos s,\ 2\sin s,\ 0\rangle$$

$$= 4\cos s \sin s + 4\cos s \sin s$$

$$= \boxed{8\cos(s) \cdot \sin(s)}$$. integrand.

(4) do the integral:

$0 \le s \le 2\pi$

$0 \le t \le 3$

$$\int_0^{2\pi} \int_0^3 8\cos(s)\sin(s)\,dt\,ds$$

$$= \int_0^{2\pi} 24\cos(s)\sin(s)\,ds$$

$u = \sin(s)$

$du = \cos(s)\,ds$

$$= \int 24u\,du = 12u^2 = 12\sin(s)^2 \Big|_0^{2\pi} = \boxed{0}$$

Ex $F = \langle 0, -z, y \rangle$

S is unit sphere in $\underbrace{1^{st} \text{ octant}}$.

x, y, z are all positive

$\vec{r}(s,t) = \langle \sin t \cos s, \sin t \sin s, \cos t \rangle$

$0 \leq t \leq \pi/2 \leftarrow \varphi$

$0 \leq s \leq \pi/2 \leftarrow \theta$

$\vec{r}_s = \langle -\sin t \sin s, \sin t \cos s, 0 \rangle$

$$\vec{r}_t = \langle \cos t \cos s,\ \cos t \sin s,\ -\sin t \rangle$$

$$\vec{r}_s \times \vec{r}_t = \langle -\cos(s)\sin^2 t,\ -\sin(s)\sin^2 t,\ -\cos t \sin t \rangle$$

$$\vec{F}(\vec{r}(s,t)) = \langle 0, -\cos t, \sin s \sin t \rangle$$

$$\langle 0, -\cos t, \sin(s)\sin(t) \rangle \cdot \langle -\cos(s)\sin^2(t),\ -\sin(s)\sin^2(t),\ -\cos t \sin(t) \rangle$$

$$= 0 + + \cos t \sin s \sin^2 t - \sin(s)\sin(t)\cos(t)\sin(t)$$

$$= 0$$

So...

$$\iint_S \vec{F}\cdot\vec{n}\,dA = \int_0^{\pi/2}\int_0^{\pi/2} 0\,ds\,dt = \boxed{0}$$