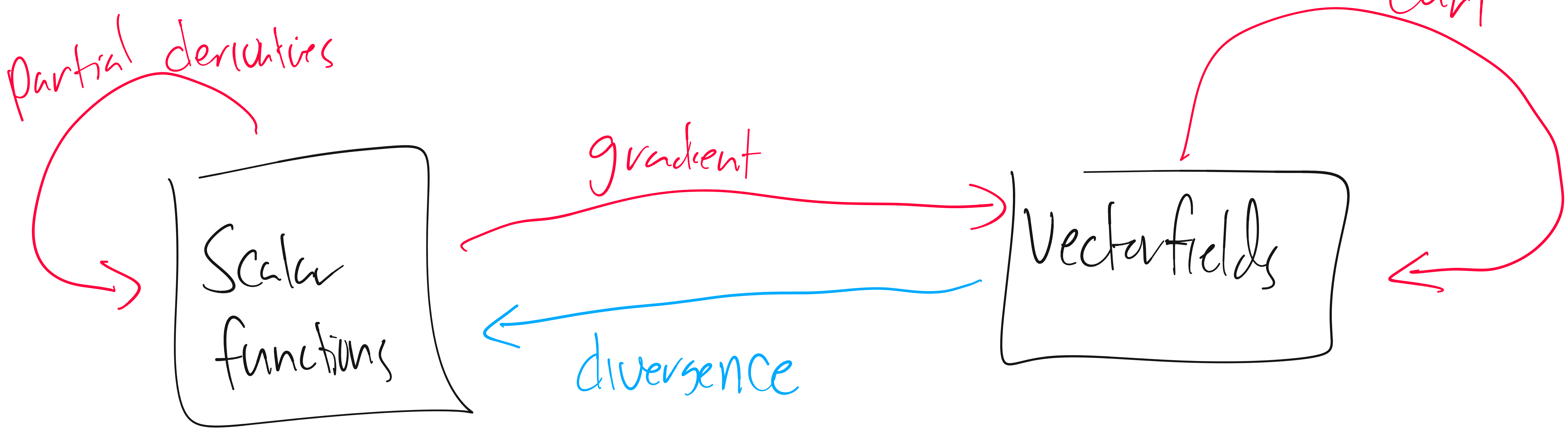
AC 12.6 // OS 6.5 | The Divergence of a Vector Field

Exam 3 is 1 week from today.

No content *after* today will be on Ex 3.



Def: $\vec{F}$: Vector field $F = \langle P, Q, R \rangle$

divergence of $\vec{F}$, denoted either as

$\operatorname{div}(\vec{F})$ or $\boxed{\nabla \cdot \vec{F}}$ is the scalar function

$$\operatorname{div}(\vec{F})(u) = \frac{\partial P}{\partial x}(u) + \frac{\partial Q}{\partial y}(u) + \frac{\partial R}{\partial z}(u)$$

where u is a point in $\mathbb{R}^3$.

$$\operatorname{div}(F) = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \cdot \langle P, Q, R \rangle$$

Curl : Cross product :: div : dot product.

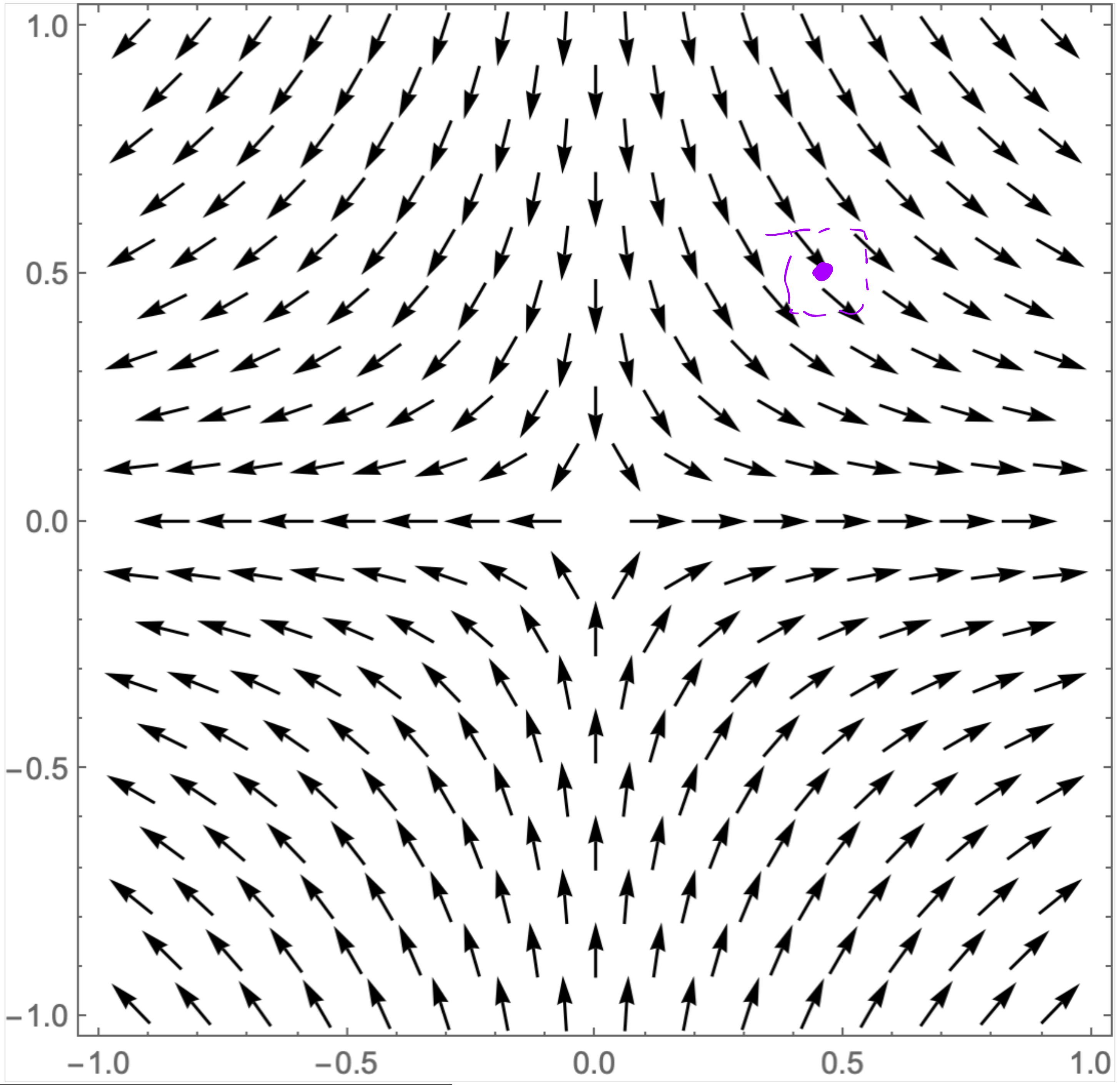
<u>Q</u> What does div measure?

Roughly speaking, $\operatorname{div}(F)$ measures how much

$\vec{F}$ "flows through" a point u

think of $\vec{F}$ as the velocity field of a fluid.

$F = \langle x, -y \rangle$



u

$\mathrm{div}(\vec{F})(u) > 0$

more fluid going out than comes in.

"Source"

$\mathrm{div}(F)(u) < 0$

more fluid coming in than going out.

"Sinks"

$\boxed{\mathrm{div}(F)(u) = 0}$

Equal in as out.

Def'n / terminology

A vfield $\vec{B}$ is called Source-free, div-free or magnetic if

$$\operatorname{div}(\vec{B}) = 0 \text{ @ every point } U.$$

Ex| $F = \langle x, -y \rangle$ is magnetic / Source-free

$$\operatorname{div}(F) = P_x + Q_y = 1 - 1 = 0$$

So $\vec{F}$ is Source-free.

Ex| $\vec{F} = \langle -y, z, x \rangle$

Is $\vec{F}$ a Source-free v-field?

$$\operatorname{div}(\vec{F}) = P_x + Q_y + R_z = 0 + 0 + 0 = 0$$

Ex) $\vec{G} = \langle 4xyz, y^2z, yz^2 \rangle$

$\underset{P}{\uparrow}$ $\underset{Q}{\uparrow}$ $\underset{R}{\uparrow}$

$\text{div}\,\vec{G} = 8yz.$

is not everywhere zero

$\Rightarrow$ not source-free.

Scratch work:

$P_x = 4yz$

$Q_y = 2yz$

$R_z = 2yz.$

Ex $\vec{F} = \nabla f$ where f admits second-order partials.

Compute $\operatorname{div}(\vec{F})$ symbolically

$$\vec{F} = \langle f_x, f_y, f_z \rangle$$

$$\operatorname{div}(\vec{F}) = P_x + Q_y + R_z = (f_x)_x + (f_y)_y + (f_z)_z$$

$$\boxed{= f_{xx} + f_{yy} + f_{zz}.}$$

Def'n: the <u>Laplacian</u> of a scalar function f is denoted

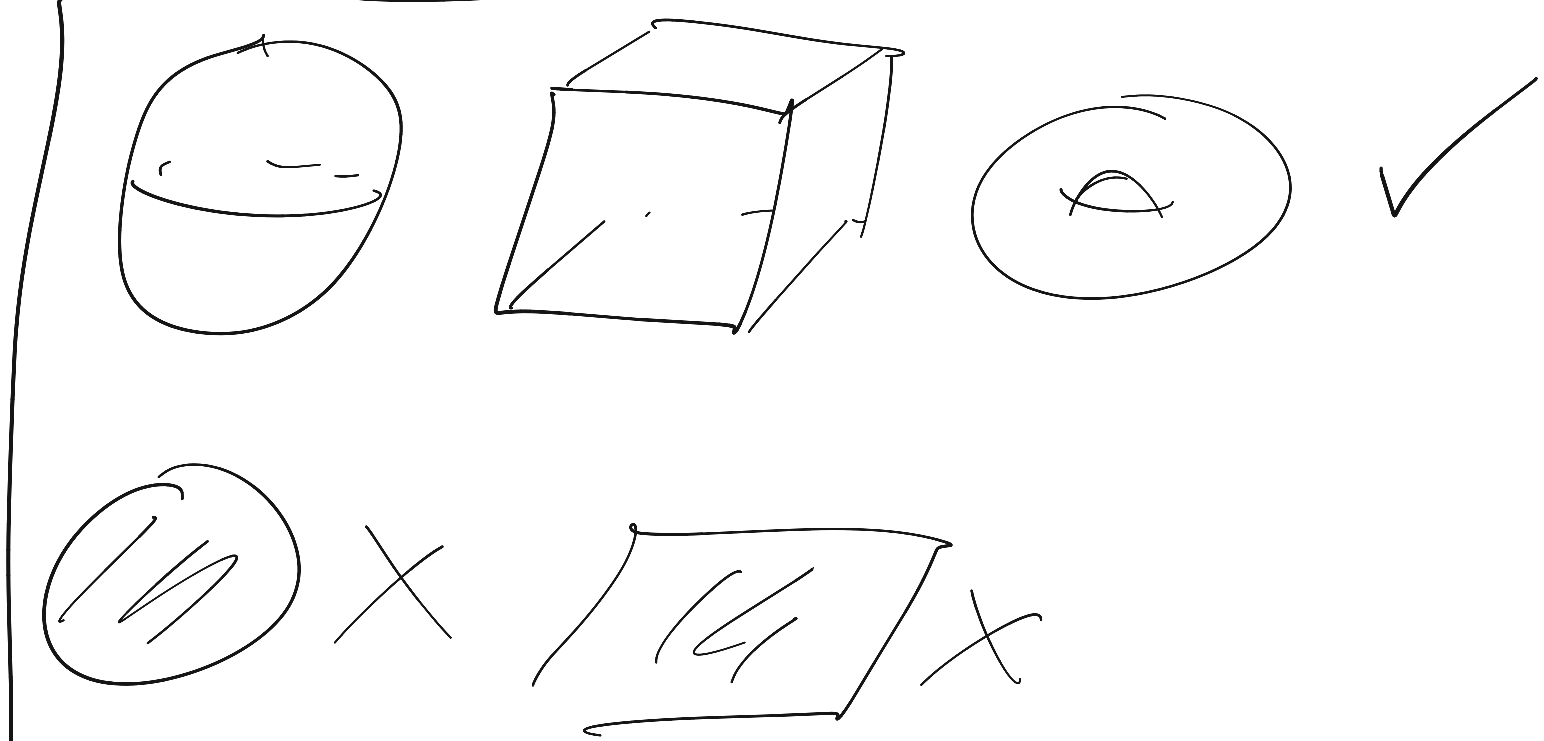
$$\Delta f := \mathrm{div}(\vec{\nabla} f)$$
$$= f_{xx} + f_{yy} + f_{zz}.$$

Closed surfaces enclose volumes.

Thm: let S be small closed surface centered @ a point p in $\mathbb{R}^3$.

Then, the flux of $\vec{F}$ through S is approximately

$$\text{flux} \approx \mathrm{div}(F)(p) \cdot \mathrm{Vol}(S)$$



arrows in
— arrows out

flux $\approx$ div(F)(p) Vol(S).

Ex (WW 12.6 #2)

flux of F through a sphere of radius 0.025

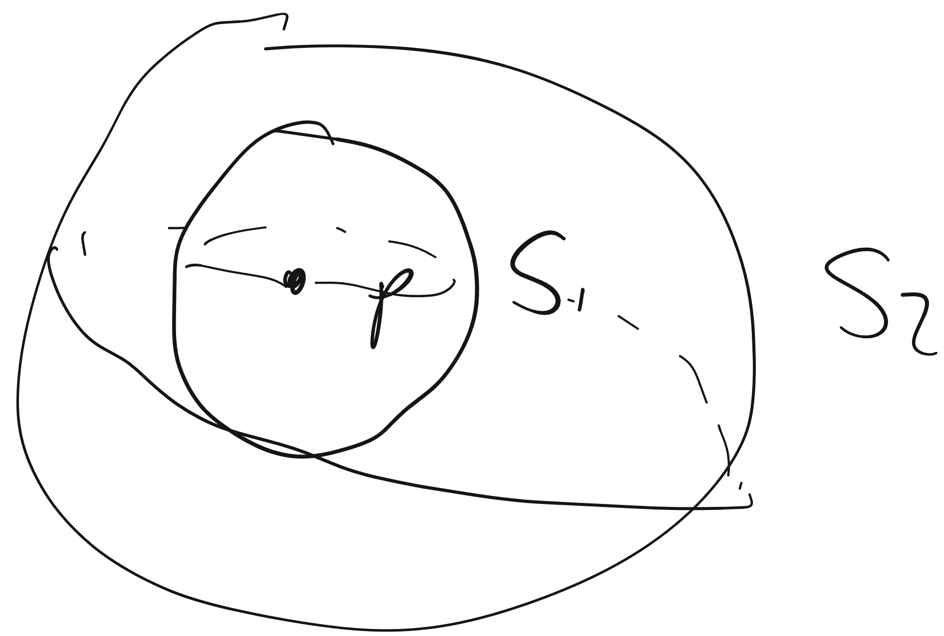
is 0.0075.

① Estimate the div(F)(p) where p is center of the sphere S.

$$\text{flux} \approx \text{div}(F)(p) \cdot \text{Vol}(S) \Rightarrow \text{div}(F)(p) \approx \frac{\text{flux}}{\text{Vol}(S)}$$

$$\operatorname{div}(F)(p) \approx \frac{0.0075}{\frac{4}{3}\pi(0.025)^3} = \boxed{114.592}$$

(2) Now, Est. flux of F through sphere of radius 0.05 ctd @ p.



$$\text{flux} \approx \text{div} \cdot \text{Vol}$$

$$\text{flux} \approx 114.592 \times \frac{4}{3}\pi(0.05)^3 \approx 0.06$$

Sphere = 2-D surface of a ball

ball = 3-D volume enclosed by a sphere