AC 12.11 // OS 6.7 : Stokes' Theorem

Exam 3 on Weds. Review tomorrow.

Def'n: A surface w/ boundary is a surface S oriented w/ normal vectors $\vec{n}$ that has an oriented boundary curve C.

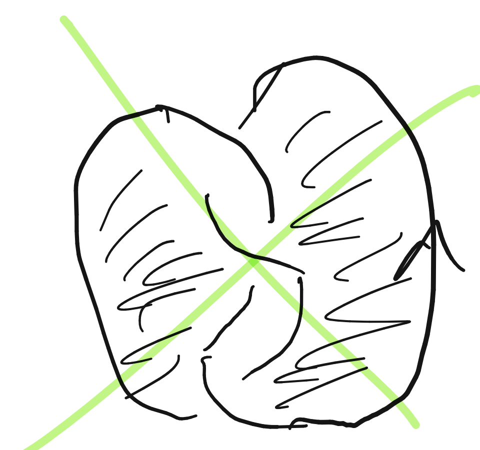
oriented w/ R.H.R

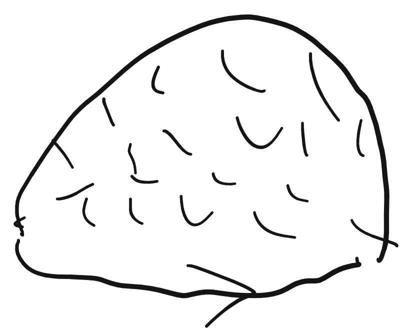
Notation: $C = \partial S$ ← "boundary"

This is the 3D version of a simple closed curve in $\mathbb{R}^2$ bounding a simply conn. region of $\mathbb{R}^2$.

<u>Theorem:</u> (Hans Seifert 1900s)

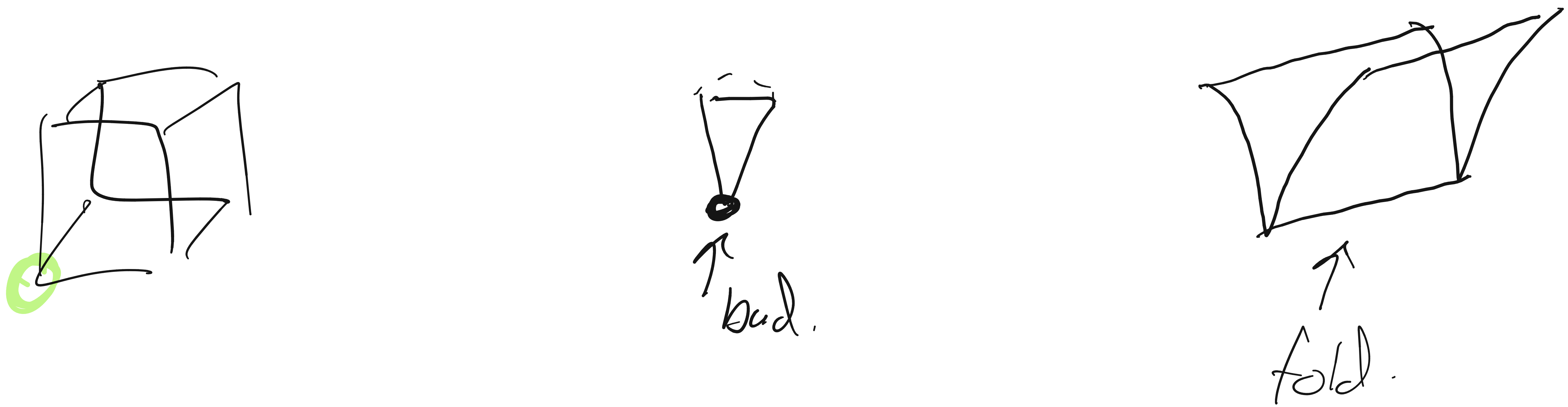
Every closed curve C in $\mathbb{R}^3$ bounds some surface in $\mathbb{R}^3$.





Idea: find a surface whose boundary is your curve.

A Surface S is <u>Smooth</u> if it has no "Corners", no "Cone point" and "no folds".



A Surface S is <u>Piece-wise Smooth</u> if it's Smooth except for finitely many folds & corners.

Recall: $\boxed{\text{Green's Theorem:}}$

C simple cl. curve in $\mathbb{R}^2$, bounds a closed region R.

F is smooth v. field on R

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_R \text{curl}(F)\, dA$$

Thm (Stokes' theorem)

F: Smooth v-field in $\mathbb{R}^3$

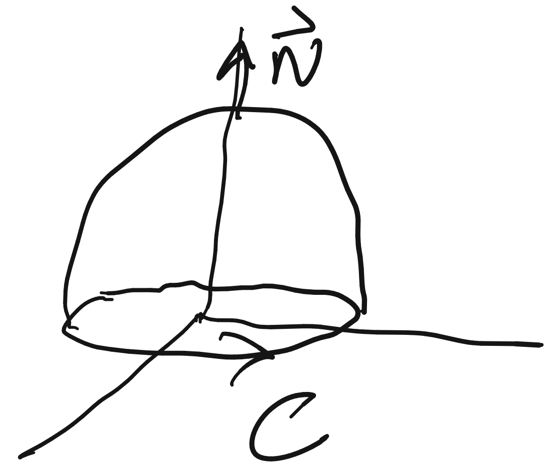
S: Smooth surface w/ bounding curve $C = \partial S$

Then:

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_S \overrightarrow{\text{curl}}(F) \cdot \vec{n} \, dA$$

flux / surface integral

Ex 105 (6.7 #327)



$F = \langle z, x, y \rangle$

S is hemisphere $z = \sqrt{a^2 - x^2 - y^2}$ $a > 0$.

$C = \partial S$ is the circle $x^2 + y^2 = a^2$ in xy-plane.

Verify Stokes' Thm by computing both

① $\oint_C \vec{F} \cdot d\vec{r}$ and $\iint_S \vec{\text{curl}}(\vec{F}) \circ \vec{n} \, dA$

C is param'd by $\vec{r}(t) = \langle a\cos t, a \cdot \sin t, 0 \rangle$

$$\vec{F} = \langle z, x, y \rangle$$

$$\vec{F}(\vec{r}(t)) = \langle 0, a \cdot \cos t, a \cdot \sin t \rangle$$

$$\vec{r}'(t) = \langle -a \sin t, a \cos t, 0 \rangle$$

$$\vec{F}(\vec{r}(t)) \cdot \vec{r}'(t) = a^2 \cos^2 t$$

IBP, u-sub

$$\oint_C \vec{F} \cdot d\vec{r} = \int_0^{2\pi} a^2 \cos^2 t \, dt = \boxed{a^2 \pi}$$

② $\iint_S \vec{\text{curl}}(\vec{F}) \cdot \vec{n}\, dA$

$\vec{r}(s,t) = \langle s, t, \sqrt{a^2 - s^2 - t^2} \rangle$

$0 \le s^2 + t^2 \le a^2$.

Domain

$\vec{F} = \langle z, x, y \rangle$.

$$\vec{\text{Curl}}(F) = \begin{vmatrix} \hat{i} & \hat{j} & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ z & x & y \end{vmatrix} = \hat{i}(1-0) - \hat{j}(0-1) + \hat{k}(1-0)$$

$$= \langle 1, 1, 1 \rangle.$$

$$\vec{n} = \vec{r}_s \times \vec{r}_t = \left\langle 1, 0, \frac{-s}{\sqrt{a^2-s^2-t^2}} \right\rangle \times \left\langle 0, 1, \frac{-t}{\sqrt{a^2-s^2-t^2}} \right\rangle$$

$$\vec{n} = \left\langle \frac{s}{\sqrt{a^2-s^2-t^2}}, \frac{t}{\sqrt{a^2-s^2-t^2}}, 1 \right\rangle$$

$$\iint_S \operatorname{curl}(\vec{F}) \cdot \vec{n} \; dA = \iint_S \langle 1,1,1 \rangle \cdot \langle \text{''} \quad \text{''} \rangle \, dA$$

$$= \int_{-a}^{a} \int_{-\sqrt{a^2-t^2}}^{+\sqrt{a^2-t^2}} \frac{s+t}{\sqrt{a^2-s^2-t^2}} + 1 \; ds\, dt$$

use polar coords: $s = r\cos\theta$ $\quad 0 \le r \le a$

$t = r\sin\theta$ $\quad 0 \le \theta \le 2\pi$

$$= \int_0^a \int_0^{2\pi} \left(\frac{r\cos\theta + r\sin\theta}{\sqrt{a^2 - r^2}} + 1 \right) r \, d\theta \, dr = a^2 \pi$$