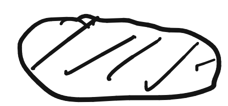
AC 12.12 / OS 6.8 || The Divergence Theorem

Recall: a surface S is closed if it has no boundary.

(opposite to a surface w/ boundary, b/c a closed surf has no bdy.)

Non-Ex  ← disk is not closed.

Ex

Thm: Every closed, orientable surface in $\mathbb{R}^3$ encloses a volume. Q

"Proof": Idea is every closed orientable surf. in $\mathbb{R}^3$ is "similar" to a sphere, donught, or a donut w/ multiple holes

all of the above shapes enclose volumes, so the original shape must've enclosed a volume.

Thm (Divergence theorem / Gauss' Thm)

S: Closed surface in $\mathbb{R}^3$,

Q: Volume enclosed by S

F: Smooth v-field on Q.

Then: $\boxed{\text{flux}} = \iint_S \vec{F} \cdot \vec{n} \, dA = \boxed{\iiint_Q \operatorname{div}(\vec{F}) \, dV}$

Note: S small surf. Q volume enc by S

$\text{flux} \approx \operatorname{div}(F)(p) \cdot \operatorname{Vol}(Q)$ p is any point in Q.

Ex find the flux through the Surface of the cuboid $0 \le x \le 2$, $1 \le y \le 4$, $0 \le z \le 1$ of the v-field $F = \langle \underbrace{x^2+yz}, y-z, 2x+2y+2z \rangle$.

don't do: Six separate flux integrals, one for each face. ☺

Instead: appeal to Div. thm:

$$\text{flux} = \iiint_Q \text{div}(F)\, dV$$

$\text{div}(F) = 2x + 1 + 2$
$= \boxed{2x+3}$.

$$\text{flux} = \iiint_Q \operatorname{div}(F)\, dV = \iiint_Q (2x+3)\, dV$$

$$= \int_0^2 \int_1^4 \int_0^1 (2x+3)\, dz\, dy\, dx$$

$$= 30$$

Ex If F is a Source-free v-field

($\mathrm{div}(F)(p) = 0$ for all points p)

then for any $\boxed{\text{closed}}$ surface S,

the flux of F through S is zero.

Proof: $\text{flux} = \iiint_Q \mathrm{div}(F)\, dV = \iiint_Q 0\, dV = 0.$

(by Div. thm.)

Ex If $\vec{F}$ v field w/ constant divergence: Ex $F = \langle 2x, 3y, 4z \rangle$

ie $\left[\operatorname{div}(F)(p) = K\right]$ for all points p.

then flux of F through any closed surf S

is $= K \cdot \operatorname{Vol}(Q)$

Proof: $\text{flux} = \iiint_Q \operatorname{div} F \, dV = \iiint_Q K \, dV = K \iiint_Q dV$

$= K \cdot \operatorname{Vol}(Q)$

$\underline{\text{Ex}}$ S closed cylinder of height 4, radius 1 w/ bottom face on xy-plane centered on z-axis

Goal: find flux of

$$F = \langle \underbrace{xy-2z}, \underbrace{y^2-yz}, 3x+z^2 \rangle$$

through S.

$$\text{div}(F) = y + 2y - z + 2z = \boxed{3y+z}$$

$$\text{flux} = \iiint_Q (3y+z)\,dV$$

$y = r\sin\theta$

$z = z$

$x = r\cos\theta$

$dV = r\,dr\,d\theta\,dz$

$$= \int_0^4 \int_0^{2\pi} \int_0^1 (3r\sin\theta + z)\, r\, dr\, d\theta\, dz = \boxed{8\pi}$$