10.1 Limits

In this section, we will explore how limts work for functions of more than one variable.

Definition: Limit of Multivariable Functions

Let $f(x,y)$ be a function of two variables defined on some region $\Omega \subseteq \mathbb{R}^2$, and let (a,b) be a point containd within Ω. We say that f **has a limit** L **as** (x,y) **approaches** (a,b) provided that, no mater what error $\varepsilon > 0$ we pick, there is some tolerance term $\delta > 0$ such that whenever $\sqrt{(x-a)^2 + (y-b)^2} \leq \delta$ then $|f(x,y) - L| \leq \varepsilon$, and we write $\lim_{(x,y)\to(a,b)} f(x,y) = L$ in this case.

We say that a function $f(x,y)$ is **continuous** at (a,b) if $f(x,y)$ is defined at (a,b), the limit $\lim_{(x,y)\to(a,b)} f(x,y)$ exists and $\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$.

There's a different way to do limits in multiple dimensions, the idea is to approach the point (a,b) from a variety of directions. The natural way to do this is to use paths/curves!

Limits along Paths

Let $f(x,y)$ be a function of two variables defined on some region $\Omega \subseteq \mathbb{R}^2$. Then, we have that $\lim_{(x,y)\to(a,b)} f(x,y) = L$ if and only if, for *any* continuous path $\gamma : (-1,1) \to \Omega$ with $\gamma(0) = (a,b)$, we have that $\lim_{t\to 0} f(\gamma(t)) = L$. [1]

It is not sufficient to check one path, one must check all paths. However, if one finds two paths γ_1 and γ_2 such that the limit of f along the γ curves are different, then you may conclude that the limit of f as (x,y) approaches (a,b) does not exist.

[1] Here that we are taking the limit of a one variable function, $g(t) = f(\gamma(t))$, so we can apply the same methods that we did in single-variable calculus!

Question 1. Consider the function $f(x,y) = \frac{2xy}{x^2+y^2}$.

(a) Compute the limit of $f(x,y)$ at the point $(0,0)$ along the lines $y = x$.
(**Hint:** One possible parameterization is $\gamma(t) = (t,t)$.)

(b) Compute the limit of $f(x,y)$ at the point $(0,0)$ along the line $y = -x$.
(**Hint:** One possible parameterization is $\gamma(t) = (-t,t)$.)

(c) What can you conclude about $\lim_{(x,y)\to(0,0)} f(x,y)$?

Question 2. For each of the following, compute the limit or show why it does not exist:

(a) $\displaystyle\lim_{(x,y)\to(2,0)} \frac{x^2y^3 - 4y^3}{xy^3 - 2y^3}$

(b) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{4x^2 + 10y^2 + 4}{4x^2 - 10y^2 + 6}$

(c) $\displaystyle\lim_{(x,y)\to(2,5)} \sqrt{\frac{1}{xy}}$

(d) $\displaystyle\lim_{(x,y)\to(1,0)} \frac{x^2 - 2xy + y^2}{x - 1}$

10.2 First Order Partial Derivatives

In this section, we will ...

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Question 1. For each of the following, compute the partial derivative specified using only the limit definition.

(a) $\frac{\partial f}{\partial x}$ for $f(x,y) = x^2 + 3xy + y^2$

(b) $\frac{\partial f}{\partial y}$ for $f(x,y) = x^2 + 3xy + y^2$

(c) $\frac{\partial f}{\partial x}$ for $f(x,y) = \frac{1}{1+x^2+y^2}$

Question 2. For each of the following, compute the partial derivative specified (you may use properties of derivatives)

(a) $\frac{\partial}{\partial x}(\sin(3x)\cos(3y))$

(d) $f_x(2,-2)$ where $f(x,y)=\frac{xy}{x-y}$.

(b) $\frac{\partial}{\partial x}\left(x^8 e^{3y}\right)$

(e) $\frac{\partial}{\partial y}\left(x^8 e^{3y}\right)$

(c) $\frac{\partial}{\partial x}\left(e^{-1/(1-x^2-y^2)}\right)$

(f) $\frac{\partial}{\partial y}\left(e^{-1/(1-x^2-y^2)}\right)$

Question 3. **Application:** The ideal gas law says, that for $n$mol of an "ideal gas", its temperature T (measured in Kelvin), pressure P (Nm^{-2}), and volume V (m^3) are related by the equation $PV = nRT$, where $R = 8.314\mathrm{Jmol}^{-1}\mathrm{K}^{-1}$. This gives us three functions, $P = P(T, V)$, $V = V(T, P)$, and $T = T(P, V)$. Use the ideal gas law to solve each of the following questions.

(a) Find $\frac{\partial P}{\partial V}$

(b) Find $\frac{\partial V}{\partial T}$

(c) Find $\frac{\partial T}{\partial P}$

(d) Use the last three parts to compute $\dfrac{\partial P}{\partial V}\dfrac{\partial V}{\partial T}\dfrac{\partial T}{\partial P}$.*

*This result is known as the "Cyclic Derivative Theorem"', and holds in general whenever you have an implicit equation of the form $f(x, y, z) = 0$.

Question 4. **Challenge Problem:** Consider the function

$$f(x,y) = \begin{cases} e^{\frac{-1}{1-\left(x^2+y^2\right)}} & 0 \leq x^2+y^2 < 1 \\ 0 & \text{otherwise} \end{cases}$$

(a) Compute $f_x(x,y)$ and $f_y(x,y)$ for points inside the open unit disk $0 \leq x^2+y^2 < 1$.

(b) Compute $f_x(x,y)$ and $f_y(x,y)$ for points outside the open unit disk, that is, when $x^2+y^2 > 1$.

(c) At the boundary of these two regions is the unit circle $x^2+y^2 = 1$. Use your answers to parts (a) and (b) to show that $f_x(x,y)$ and $f_y(x,y)$ are continuous at any point on the unit circle. (Hint: What value do $f_x(x,y)$ and $f_y(x,y)$ approach as (x,y) approach a boundary point from the inside of the unit disk?)