10.8 Constrained Optimization: Lagrange Multipliers

Technique: Lagrange Multipliers

To minimize/maximize a funciton $f(\mathbf{x})$ with respect to the constraint $g(\mathbf{x}) = c$, we setup an auxillary function, $\mathcal{L}(\mathbf{x}, \lambda)$ called the **Lagrangian**, defined as $\mathcal{L}(\mathbf{x}, \lambda) = f(\mathbf{x}) - \lambda(g(\mathbf{x}) - c)$. The newly introduced variable λ is called a **Lagrange multiplier**, and is the namesake of this technique.

To find optimal values of $\mathbf{x}$, and λ that satisfy these conditions, we want to solve $\boldsymbol{\nabla}\mathcal{L}(\mathbf{x}, \lambda) = \mathbf{0}$ for the components of $\mathbf{x}$ and λ. We then plug each point $\mathbf{x}$ into f to determine minimality/maximality.

Question 1. **(Warmup)** A soup can in the shape of a right circular cylinder is to be made from two materials. The material for the side of the can costs $0.015 per square inch and the material for the lids costs $0.027 per square inch. Suppose that we desire to construct a can that has a volume of 16 cubic inches. What dimensions minimize the cost of the can?

Question 2. For each of the following setups, use the method of Lagrange multipliers to find the minimum and maximum values of the function subject to the given constraint:

(a) $f(x, y) = x^2 y$, subject to $x^2 + 2y^2 = 6$

(b) $f(x, y) = xy$, subject to $4x^2 + 8y^2 = 16$

(c) $f(x,y,z) = x^2 + y^2 + z^2$, subject to $x^4 + y^4 + z^4 = 1$

(d) $f(x,y,z) = x + y + z$, subject to $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 1$

Question 3. Find the point on the plane $x + y + z = 1$ that minimizes the distance from the origin.

Question 4. Find the minumum and maximum distance between the origin of $\mathbb{R}^2$ and the ellipse $x^2 + xy + 2y^2 = 1$.

Question 5. Find the minimum and maximum values of the function $h(x, y, z) = 3xy + 2xz - yz$ on the sphere $x^2 + y^2 + z^2 = 1$.

Question 6. The curve $x^3 - y^3 = 1$ in $\mathbb{R}^2$ is asymptotic to the line $y = x$. Find the points on this curve that are furthest from the line $y = x$.

Question 7. A shipping company handles rectangular boxes, provided that sum of the length, width, and height of the box does not exceed 120cm. Find the dimensions of the box that meets this condition and has the largest volume. How do you know that the volume you found is maximal?

Question 8. **(Challenge Problem)** We can extemd the method of Lagrange multipliers to functions of any number of variables and any number of constraints. Suppose we have some real-valued function $f(x_1, \ldots, x_n)$ that we want to optimize, subject to the contstraits $g_1(\mathbf{x}) = c_1, \ \ldots, \ g_m(\mathbf{x}) = c_m$. We can define the Lagrangian as $\mathcal{L}(\mathbf{x}, \lambda_1, \ldots, \lambda_m) = f(\mathbf{x}) - \sum_{i=1}^{m} \lambda_i(g_i(\mathbf{x}) - c_i)$, where the λ_i are the Lagrange multipliers. To optimize, we solve the equations $\boldsymbol{\nabla}\mathcal{L} = \mathbf{0}$ for the x_i and λ_j. We then plug each point $\mathbf{x} = (x_1, \ldots, x_n)$ into f to determine minimality/maximality.

(a) Minimize the function $f(x, y, z) = x^2 + y^2 + z^2$ subject to the constraints $x + y + z = 9$ and $2xy - z = 1$. Use Lagrange multipliers to find the points (x, y, z) that minimze the funciton f.

(b) Maximize the function $f(x, y, z) = 1000 - x^2 - y^2 - z^2$ subject to the constraints $2x + y + 23z = 9$ and $5x + 5y + 7z = 29$ using the method of Lagrange multipliers.