9.1 Functions of Several Variables and Three Dimensional Space

9.1.1 Functions of Several Variables and Three Dimensional Space - Part I

In this section, we will introduce functions of two variables

Definitions

A **function** f **of two independent variables** is a rule which assigns to each ordered pair (x, y) in a set D exactly one real number output, $f(x, y)$.

The **domain** of a function f is the set of all inputs at which the function is defined.

Example

The function $f(x, y) = x + y$ is a function of two variables. For the input $(1, 2)$, the function f assigns the output 1+2=3, or $f(1, 2) = 2$. As the function is defined for any ordered pair (x, y), we say that the domain of $f(x, y)$ is the entire xy-plane, or $\mathbb{R}^2$.

Question 1. For each of the following functions of two variables determine if the function is defined at $(1, 2)$. If it is, evaluate $f(1, 2)$. If it is not, explain why.

(a) $f(x, y) = x^2 + y^2$

(b) $g(x, y) = \frac{x}{y}$

(c) $h(x, y) = \sqrt{x^2 + y^2}$

(d) $j(x, y) = \frac{x+1}{y-2}$

(e) $k(x, y) = 0$

(f) $l(x, y) = 2x$

Question 2. For each function in Question 1, state the domain.

9.1.2 Functions of Several Variables and Three Dimensional Space - Part II

In this section, we will introduce graphs of two variable functions

Definition

The **graph** of a function $z = f(x, y)$ is the set of points of the form $(x, y, f(x, y))$, where the ordered pair (x, y) is in the domain of f.

Important Examples

$z = x^2 + y^2$
$z = \sqrt{x^2 + y^2}$
$z = x^2 - y^2$

Some Transformations of Graphs

Let G be the graph of the function $z = f(x, y)$.

1. The the graph of the function $z = -f(x, y)$ is G inverted over the xy-plane.
2. The graph of the function $z = f(x - a, y)$ is G shifted a units in the x-direction.
3. The graph of the function $z = f(x, y - b)$ if G shifted b units in the y-direction.
4. The graph of the function $z = k \cdot f(x, y)$, where $k > 0$, is f scaled vertically by the factor k.

Question 1. Using the above important examples and listed transformations, sketch graphs of the following functions:

(a) $f(x, y) = x^2 + y^2$

(b) $g(x, y) = \frac{x}{y}$

(c) $h(x, y) = \sqrt{x^2 + y^2}$

(d) $j(x, y) = \frac{x+1}{y-2}$

(e) $k(x, y) = 0$

(f) $l(x, y) = 2$

9.1.3 Functions of Several Variables and Three Dimensional Space - Part III

In this section, we will introduce traces, contours, and add to our library of graphable functions.

Definitions

A **trace** of a function of two independent variables, $z = f(x, y)$, in the x direction is a curve of the form $z = f(x, c)$, where c is a constant. Likewise, a **trace** of a function of two independent variables, $z = f(x, y)$, in the y direction is a curve of the form $z = f(c, y)$, where c is a constant.

A **contour (or level curve)** of a function $z = f(x, y)$ is a curve of the form $k = f(x, y)$, where k is a constant.

Question 1. For the following functions, sketch the trace of the function in the x direction at $c = 2$. Sketch the trace of the function in the y direction at $c = 1$.

(a) $f(x, y) = x^2 + y^2$

(b) $g(x, y) = \frac{x}{y}$

(c) $l(x, y) = 2$

Question 2. For the functions in Question 1, plot the contours for $k = -1$, 0, and 1.

Important Example

A function $z = f(x, y)$ is said to be **radially symmetric** if it can be expressed in the form $z = h(r)$, where $r = \sqrt{x^2 + y^2}$ as in polar coordinates. In other words, the output of the function $f(x, y)$ at a point (x, y) depends only on the distance $r = \sqrt{x^2 + y^2}$ of the point from the origin in the xy-plane.

Radially symmetric functions can be sketched quite easily. First graph the trace in the y direction at $c = 0$ on the yz-plane. Then rotate that graph about the z-axis.

Question 3. Consider the function $f(x, y) = \left(x^2 + y^2\right)^2$. Notice that $f(x, y) = \left(r^2\right)^2 = r^4$, where $r = \sqrt{x^2 + y^2}$. Thus $f(x, y)$ is radially symmetric. Sketch a graph of $f(x, y)$.

Question 4. Sketch a graph of $g(x, y) = \sin\left(\sqrt{x^2 + y^2}\right)$.

Question 5. (a) Which of the examples from the beginning of the 9.1 - Part II worksheet are radially symmetric?

(b) What can you say about the contours of a radially symmetric function?