9.3 The Dot Product and Projections

In this section, we look at the definition and uses of the dot product

Definition and Properties

Algebraically, the dot product of two vectors, $\mathbf{v} = a\hat{\mathbf{i}} + b\hat{\mathbf{j}} + c\hat{\mathbf{k}}$ and $\mathbf{w} = d\hat{\mathbf{i}} + e\hat{\mathbf{j}} + f\hat{\mathbf{k}}$, is

$$\mathbf{v} \cdot \mathbf{w} := ad + be + cf.$$

Geometrically, the dot product of $\mathbf{v}$ and $\mathbf{w}$ is the product of their lengths and the cosine of the angle between them:

$$\mathbf{v} \cdot \mathbf{w} := \|\mathbf{v}\|\|\mathbf{w}\| \cos\theta$$

where θ is the angle between the two vectors.

The most important property of the dot product is:

$$\mathbf{v} \cdot \mathbf{w} = 0 \text{ if and only if } \mathbf{v} \text{ is perpendicular to } \mathbf{w}.$$

Question 1. Compute the dot product of $3\hat{\mathbf{i}} + 5\hat{\mathbf{j}} - \hat{\mathbf{k}}$ and $2\hat{\mathbf{i}} - 3\hat{\mathbf{j}} + 4\hat{\mathbf{k}}$. What is the angle between these vectors?

Question 2. What $\mathbf{v} \cdot \mathbf{v}$? (What is the angle of between a vector and itself?)

Question 3. Is the vector from $(4, 1, 2)$ to $(2, 4, 3)$ perpendicular to the vector $3\hat{\mathbf{i}} - 2\hat{\mathbf{j}} + 4\hat{\mathbf{k}}$?

Question 4. The vectors $3\hat{\mathbf{i}} - 5\hat{\mathbf{j}} + 2\hat{\mathbf{k}}$ and $\hat{\mathbf{i}} - y\hat{\mathbf{j}}$ are perpendicular. Use the dot product to find y.

Question 5. Given vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ so that $\mathbf{u} \cdot \mathbf{v} = 2$, $\mathbf{u} \cdot \mathbf{w} = -3$, $\|\mathbf{u}\| = 5$, $\|\mathbf{v}\| = 1/4$, and $\|\mathbf{w}\| = 1$, compute the following:

(a) $2\mathbf{u} \cdot (\mathbf{v} + \mathbf{w})$

(b) The angle between $\mathbf{v}$ and $\mathbf{w}$.

(c) If $(2\mathbf{u} + \mathbf{v}) \cdot (2\mathbf{w}) = -6$, find $\mathbf{v} \cdot \mathbf{w}$.

Projection of one vector onto another

Given a vector $\mathbf{v}$, we want to know 'how much' of another vector $\mathbf{u}$ points in the direction of $\mathbf{v}$. This is called the projection of $\mathbf{u}$ onto $\mathbf{v}$ and it is a multiple of $\mathbf{u}$, as you can see in the following diagram:

Left: $\text{proj}_{\mathbf{v}}\,\mathbf{u}$, Right: $\text{proj}_{\mathbf{v}}\,\mathbf{u}$ when $\theta > \frac{\pi}{2}$

Algebraically,

$$\text{proj}_{\mathbf{v}}(\mathbf{u}) = \frac{\mathbf{v}\cdot\mathbf{u}}{\mathbf{v}\cdot\mathbf{v}}\mathbf{v} = \frac{\mathbf{v}\cdot\mathbf{u}}{\|\mathbf{v}\|^2}\mathbf{v} = \frac{\mathbf{v}\cdot\mathbf{u}}{\|\mathbf{v}\|}\hat{\mathbf{v}}$$

Question 6. Compute $\text{proj}_{\mathbf{v}}(\mathbf{u})$ for the following choices of $\mathbf{u}$ and $\mathbf{v}$:

(a) $\mathbf{u} = 3\hat{\mathbf{i}} + 5\hat{\mathbf{j}} - \hat{\mathbf{k}}$, $\mathbf{v} = 2\hat{\mathbf{i}} - 3\hat{\mathbf{j}} + 4\hat{\mathbf{k}}$.

(b) $\mathbf{u} = 3\hat{\mathbf{i}} + 5\hat{\mathbf{j}} - \hat{\mathbf{k}}$, $\mathbf{v} = -2\hat{\mathbf{i}} + 3\hat{\mathbf{j}}d - 4\hat{\mathbf{k}}$.

(c) $\mathbf{u} = 3\hat{\mathbf{i}} + 5\hat{\mathbf{j}} - \hat{\mathbf{k}}$, $\mathbf{v} = \hat{\mathbf{i}}$.

(d) $\mathbf{u} = 3\hat{\mathbf{i}} + 5\hat{\mathbf{j}} - \hat{\mathbf{k}}$, $\mathbf{v} = 5\hat{\mathbf{i}}$.