

9.3 The Dot Product and Projections

In this section, we look at the definition and uses of the dot product

Definition and Properties

Algebraically, the dot product of two vectors, $\mathbf{v} = a\hat{\mathbf{i}} + b\hat{\mathbf{j}} + c\hat{\mathbf{k}}$ and $\mathbf{w} = d\hat{\mathbf{i}} + e\hat{\mathbf{j}} + f\hat{\mathbf{k}}$, is

$$\mathbf{v} \cdot \mathbf{w} := ad + be + cf.$$

Geometrically, the dot product of \mathbf{v} and \mathbf{w} is the product of their lengths and the cosine of the angle between them:

$$\mathbf{v} \cdot \mathbf{w} := \|\mathbf{v}\|\|\mathbf{w}\|\cos\theta$$

where θ is the angle between the two vectors.

The most important property of the dot product is:

$$\mathbf{v} \cdot \mathbf{w} = 0 \text{ if and only if } \mathbf{v} \text{ is perpendicular to } \mathbf{w}.$$

Question 1. Compute the dot product of $3\hat{\mathbf{i}} + 5\hat{\mathbf{j}} - \hat{\mathbf{k}}$ and $2\hat{\mathbf{i}} - 3\hat{\mathbf{j}} + 4\hat{\mathbf{k}}$. What is the angle between these vectors?

Question 2. What $\mathbf{v} \cdot \mathbf{v}$? (What is the angle of between a vector and itself?)

Question 3. Is the vector from $(4, 1, 2)$ to $(2, 4, 3)$ perpendicular to the vector $3\hat{\mathbf{i}} - 2\hat{\mathbf{j}} + 4\hat{\mathbf{k}}$?

Question 4. The vectors $3\hat{\mathbf{i}} - 5\hat{\mathbf{j}} + 2\hat{\mathbf{k}}$ and $\hat{\mathbf{i}} - y\hat{\mathbf{j}}$ are perpendicular. Use the dot product to find y .

Question 5. Given vectors \mathbf{u} , \mathbf{v} , and \mathbf{w} so that $\mathbf{u} \cdot \mathbf{v} = 2$, $\mathbf{u} \cdot \mathbf{w} = -3$, $\|\mathbf{u}\| = 5$, $\|\mathbf{v}\| = 1/4$, and $\|\mathbf{w}\| = 1$, compute the following:

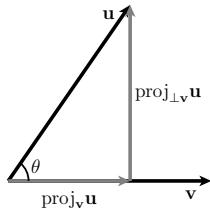
(a) $2\mathbf{u} \cdot (\mathbf{v} + \mathbf{w})$

(b) The angle between \mathbf{v} and \mathbf{w} .

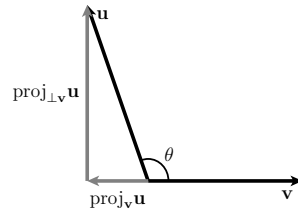
(c) If $(2\mathbf{u} + \mathbf{v}) \cdot (2\mathbf{w}) = -6$, find $\mathbf{v} \cdot \mathbf{w}$.

Projection of one vector onto another

Given a vector \mathbf{v} , we want to know ‘how much’ of another vector \mathbf{u} points in the direction of \mathbf{v} . This is called the projection of \mathbf{u} onto \mathbf{v} and it is a multiple of \mathbf{u} , as you can see in the following diagram:



Left: $\text{proj}_{\mathbf{v}} \mathbf{u}$,



Right: $\text{proj}_{\mathbf{v}} \mathbf{u}$ when $\theta > \frac{\pi}{2}$

Algebraically,

$$\text{proj}_{\mathbf{v}}(\mathbf{u}) = \frac{\mathbf{v} \cdot \mathbf{u}}{\mathbf{v} \cdot \mathbf{v}} \mathbf{v} = \frac{\mathbf{v} \cdot \mathbf{u}}{\|\mathbf{v}\|^2} \mathbf{v} = \frac{\mathbf{v} \cdot \mathbf{u}}{\|\mathbf{v}\|} \hat{\mathbf{v}}$$

Question 6. Compute $\text{proj}_{\mathbf{v}}(\mathbf{u})$ for the following choices of \mathbf{u} and \mathbf{v} :

(a) $\mathbf{u} = 3\hat{\mathbf{i}} + 5\hat{\mathbf{j}} - \hat{\mathbf{k}}$, $\mathbf{v} = 2\hat{\mathbf{i}} - 3\hat{\mathbf{j}} + 4\hat{\mathbf{k}}$.

(b) $\mathbf{u} = 3\hat{\mathbf{i}} + 5\hat{\mathbf{j}} - \hat{\mathbf{k}}$, $\mathbf{v} = -2\hat{\mathbf{i}} + 3\hat{\mathbf{j}} - 4\hat{\mathbf{k}}$.

(c) $\mathbf{u} = 3\hat{\mathbf{i}} + 5\hat{\mathbf{j}} - \hat{\mathbf{k}}$, $\mathbf{v} = \hat{\mathbf{i}}$.

(d) $\mathbf{u} = 3\hat{\mathbf{i}} + 5\hat{\mathbf{j}} - \hat{\mathbf{k}}$, $\mathbf{v} = 5\hat{\mathbf{i}}$.