9.6 Vector-Valued Functions

In this section, we will begin exploring functions whose codomain have dimension larger than 1.

Main Concepts

A **vector-valued function** is a function whose input is a single real number t, and whose output is a a vector that depends on t. The **graph** of a vector-valued function is the set of all terminal points of the output vectors with their initial point at the origin. A **parametric curve** (or **parametric equation**) is a vector-valued function of the form

$$\mathbf{r}(t) = x(t)\hat{\mathbf{x}} + y(t)\hat{\mathbf{y}} \qquad \text{or} \qquad \mathbf{r}(t) = x(t)\hat{\mathbf{x}} + y(t)\hat{\mathbf{y}} + z(t)\hat{\mathbf{z}}$$

where the **component functions** $x(t)$, $y(t)$, and $z(t)$ are real-valued functions.

A **parametric line** is a vector-valued function $\mathbf{r}(t)$ of the form $\mathbf{r}(t) = \mathbf{b} + t\mathbf{v}$, where $\mathbf{b}$ and $\mathbf{v}$ are fixed vectors. We also say that this is a line **starting at b, in the direction of v**.

Question 1. In this problem, we will generalize the process of finding the equation of a 2-dimensional line to the process of finding the (parametric) equation of a line in higher dimensions.

(a) Let (x_0, y_0) and (x_1, y_1) be two points in $\mathbb{R}^2$, and assume that $x_0 \neq x_1$. Find a linear function that passes through these points. (**Hint:** point-point form)

(b) Now, (x_0, y_0) and (x_1, y_1) be two points in $\mathbb{R}^2$, but don't assume that $x_0 \neq x_1$. We want to find a parametric function $\mathbf{r}(t) = x(t)\hat{\mathbf{x}} + y(t)\hat{\mathbf{y}}$ such that $\mathbf{r}(0) = (x_0, y_0)$ and $\mathbf{r}(1) = (x_1, y_1)$.

Suppose $x(t) = a + bt$ and $y(t) = c + dt$. Then, we want $x(0) = x_0$ and $x(1) = x_1$ and similarly for y. Find values of a, b, c, d that satisfy the equations above.

(c) Using your function from the last part, solve $y = y(t)$ and $x = x(t)$ for t.

(d) Equate the two expressions you found in the last part to get a single equation involving only x and y. What do you notice about this equation?

(e) Do part (b) for the case where (x_0, y_0, z_0) and (x_1, y_1, z_1) are two points in $\mathbb{R}^3$.

Question 2. Find a parameterization of a circle of radius 11, contained in the plane $z = 0$ centered at the point $(1, -1, 0)$, and oriented clockwise.

Question 3. Find a vector-valued function $\mathbf{r}(t)$ that parameterizes the line through the point $(-2, 1, 4)$ in the direction of the vector $\mathbf{b} = \langle 3, 2, -5 \rangle$.

Question 4. Find a vector-valued function $\mathbf{r}(t)$ that parameterizes the line of intersection of the planes $x + 2y - z = 4$ and $3x + y - 2z = 1$.

Question 5.

(a) Determine the point of intersection of the lines given by $\mathbf{r}(t) = \langle 2, 1, 0\rangle + \langle 1, -2, 4\rangle t$ and $\mathbf{s}(t) = \langle 3, 3, 0\rangle + \langle 1, -2, 2\rangle t$.

(b) Then, find a vector valued function $\mathbf{q}(t)$ that parameterizesthe line that passes through the point of intersection and is perpendicular to the lines traced out by $\mathbf{r}(t)$ and $\mathbf{s}(t)$.